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Tome XX (XXIV), Fasc. 1—2 1974

Secţia I

MATEMATICĂ. MECANICĂ TEORETICĂ. FIZICĂ

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D.C.00752

ON A FORMALIZATION OF ABSTRACT MODULAR NETWORKS

BY

CORINA REISCHER and DAN SIMOVICI

1. Introduction

Modular networks are models useful both for automata theory and for the study of neural networks. These two points of view are represented by M. A. Arbib [1], [2] and respectively McCullogh and Pitts [3], which have used modular networks described in a physically manner.

Our aim is to present a formal approach of this notion, fit for mathematical purposes and to develop some results concerning simulation of automata by networks.

II. Modular Networks

Let I_k be a finite set $I_k = \{i_1, i_2, ..., i_k\}$.

Definition 1. A *k-device* is a 4-tuple $\mathbf{h} = \{I_k, h, \{\beta^j_{\mathbf{h}}\}_{j=\overline{1, n(h)}}, \delta_{\mathbf{h}}\}$ where $h : I_k^{n(h)}$*), $\to I_k$, $n(\mathbf{h})$ is a natural number, $\{\beta^j_{\mathbf{h}}\}_{j=\overline{1, n(h)}}$ is a set of mappings $\beta^j_{\mathbf{h}} = \mathrm{pr}_j : I_k^{n(h)} \to I_k$ called the set of input wires and $\delta_{\mathbf{h}} : h(I_k^{n(h)}) \to$ $\to I_k$ is the identity mapping of the set $h(I_k^{n(\mathbf{h})})$ called output wire.

A k — device is conceived as a „black box" with $n(\mathbf{h})$ input wires and an output wire. When $n(\mathbf{h})$ input symbols $i_{j_1}, ..., i_{j_{n(\mathbf{h})}}$ are applied to the input wires at the moment t, an output signal $h(i_{j_1}, ..., i_{j_{n(\mathbf{h})}}) \in I_k$ is generated at the output wire at the moment $t + 1$.

Remark 1. A k — device $\mathbf{h}$ can be regarded as a finite automaton $A = [I_k^{n(\mathbf{h})}, \{s\}, I_k, f, g]$, where $I_k^{n(\mathbf{h})}$ is the input alphabet of the automaton, $\{s\}$ is the set of states which consists from a singleton, I_k is the ouput alphabet, the nextstate function $f : \{s\} \times I_k^{n(\mathbf{h})} \to \{s\}$ is the constant mapping, the output function $g : \{s\} \times I_k^{n(\mathbf{h})} \to I_k$ is given by $g(s, (i_{j_1}, ..., i_{j_{n(\mathbf{h})}})) =$ $= h(i_{j_1}, ..., i_{j_{n(\mathbf{h})}})$.

---

*) $I_k^{n(\mathbf{h})} = \underbrace{I_k \times \ldots \times I_k}_{n(\mathbf{h})}$

The output symbol is delayed an unit of time.

Let $\mathbf{H} = \{\mathbf{h}_1, \mathbf{h}_2, ..., \mathbf{h}_m\}$ a set of k-devices, $B = \{\beta^1_{\mathbf{h}_1}, ..., \beta^{n\,\mathbf{h}(m)}_{\mathbf{h}_m}\}$ the set of input wires, $D = \{\delta_{\mathbf{h}_1}, ..., \delta_{\mathbf{h}_m}\}$ the set of output wires of the devices belonging to $\mathbf{H}$.

D e f i n i t i o n 2. O k — *device modular network* (shortly, a *modular network*) is a 5 — tuple $\mathbf{R} = (\mathbf{H}, B_1, \theta, Z; D_1)$ where $B_1 \subseteq B$ is the set of input wires of the network, θ is an equivalence on the set $B \setminus B_1$, z is an injective mapping, $z : (B \setminus B_1/\theta \to D$ and $D_1 \subseteq D$ is the set of output wires of the network $\mathbf{R}$. The input wires belonging to the same class of equivalence $[\beta^{\mathbf{h}}_{\mathbf{h}_j}]_\theta$ are connected to the output wire $z([\beta^{\mathbf{h}}_{\mathbf{h}_j}]_\theta) \in D$.

Remark 2. The k — device modular network $\mathbf{R}$ can be regarded as a finite automaton $A_{\mathbf{R}} = [I_k^{|B_1|}, I_k^m, I_k^{|D_1|}, f_{\mathbf{R}}, g_{\mathbf{R}}]$ where $f_{\mathbf{R}}$ $((i_1(t), ..., i_m(t)), (i'_1(t), ..., i'_{|B_1|}(t)) = (i_1(t+1), i_2(t+1), ..., i_m(t+1))$, where $i_l(t)$ is the symbol yielded at the time t by the output wire at the device $\mathbf{h}_l$, $l = \overline{1, m}$, $i_j(t)$ is the symbol applied at the j-th input wire of the network at the moment t.

We have:

$$i_l(t+1) = h_l({}^t(z([\beta^1_{\mathbf{h}_l}]_\theta)), ..., {}^t(z([\beta^{n(h_l)}_{\mathbf{h}_l}]_\theta)))$$

and

$$g_{\mathbf{R}}(\mathbf{i}_1(t), ..., i_m(t)), (i'_1(t), ..., i'_{|B_1|}(t))) = (i''_{j_1}(t+1), ..., i''_{J_{|D_1|}}(t+1)),$$

where $i''_l(t+1) = h_l({}^t z([\beta^1_{\mathbf{h}_l}]_\theta), ..., {}^t(z([\beta^{n(h_l)}_{\mathbf{h}_l}]_\theta)), l \in \{j_1, ..., j_{|D_1|}\}$.

If F is a time-dependent function, we have denoted by ${}^tF$ its value at time t.

Example. Let $\mathbf{R}$ be 2 - device modular network from fig. 1. For this network $I_2 = \{0,1\}$, $B_1 = \{\beta^1_{h_1}\}$, $D_1 = \{\delta_{h_3}\}$, $(B / B_1)/\theta = \{\{\beta^2_{h_1}, \beta^2_{h_2}\}, \{\beta^1_{h_2}\}, \{\beta^1_{h_3}\}\}$, z is given by

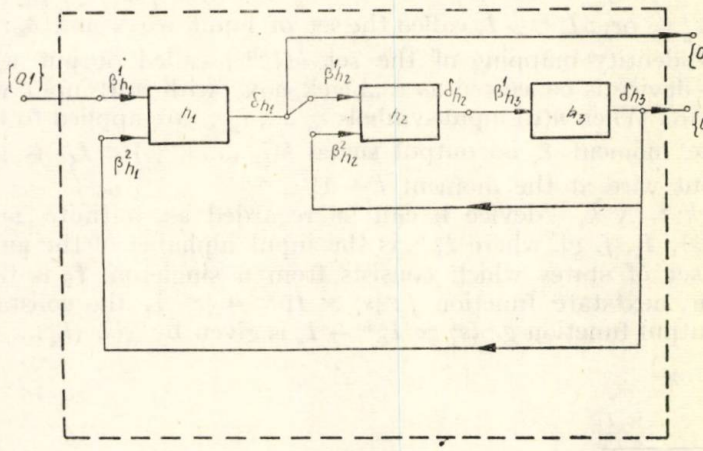

Fig. 1

z	$\{\beta^2_{h_1}, \beta^2_{h_2}\}$	$\{\beta^1_{h_2}\}$	$\{\beta^1_{h_3}\}$
	δ_{h_3}	δ_h	δ_{h_2}

and h_1, h_2, h_3 are given by:

h_1	(0,0)	(0,1)	(1,0)	(1,1)
	0	0	0	1

h_2	(0,0)	(0,1)	(1,0)	(1,1)
	1	1	1	0

h_3	0	1
	1	0

The automaton $A_{\mathbf{R}} = [I_2, I_2^3, I_2^2, f_{\mathbf{R}}, g_{\mathbf{R}}]$ is described by the table

		(0, 0, 0)	(0, 0, 1)	(0, 1, 0)	(0, 1, 1)	(1, 0, 0)	(1, 0, 1)	(1, 1,0)	(1, 1, 1)
$f_{\mathbf{R}}$	0	(0, 1, 1)	(0, 1, 1)	(0, 1, 0)	(0, 1, 0)	(0, 1, 1)	(0, 0, 1)	(0, 1, 0)	(0, 0, 0)
	1	(0, 1, 1)	(1, 1, 1)	(0, 1, 0)	(1, 1, 0)	(0, 1, 1)	(1, 0, 1)	(0, 1, 0)	(1, 0, 0)
$g_{\mathbf{R}}$	0	(0, 1)	(0, 1)	(0, 0)	(0, 0)	(0, 1)	(0, 1)	(0, 0)	(0, 0)
	1	(0, 1)	(1, 1)	(0, 0)	(1, 0)	(0, 1)	(1, 1)	(0, 0)	(1, 0)

III. Simulation of Automata by Modular Networks

Let $A_j = [K_j, S_j, O_j. f_j, g_j]$, $j = 1, 2$ two automata. A_2 simulates A_1 by the simulation $\psi = (t, u, v)$, where $t : K_1^* \to K_2^*$, (K_j^* being the free semigroup generated by the input alphabet K_j), $u : S_1 \to S_2$, $v : O_2 \to O_1$ are mappings for which the diagram

$$\begin{tikzcd} K_1^* \arrow[r, "f_{1s}"] \arrow[d, "t"'] & O_1 \\ K_2^* \arrow[r, "f_{2u(s)}"] & O_2 \arrow[u, "v"'] \end{tikzcd}$$

is commutative. Here $f_{js} : K_j^* \to O_j$, are given by $f_{js}(p) = f_j(s, p)$, $\forall s \in S_j$, $p \in K_j^*$, $j = 1, 2$.

Definition 3. The network $\mathbf{R}$ simulates the automaton A if $A_{\mathbf{R}}$ simulates A.

We have

Theorem. *For every finite automaton A there exists a modular network $\mathbf{R}_A$ which simulates A.*

P r o o f. Let $A = [K, S, O, f, g]$, where $K = \{i_1, ..., i_n\}$, $S = \{s_1, ..., s_l\}$, $O = \{O_1, ..., O_m\}$.

The modular network $\mathbf{R_A}$ which will be effectively constructed will contain lm, 2 — devices with $I_2 = \{0, 1\}$.

For each pair $(j_1, j_2) \in \{1, ..., l\} \times \{1, ..., m\}$ we shall consider a 2 - device $\mathbf{h}_{(j_1, j_2)}$ with $n(\mathbf{h}_{(j_1, j_2)}) = lm + n$.

Regarding the network $\mathbf{R_A} = (\mathbf{H}, B_1, \theta, z; D_1)$ we have:

$$\mathbf{H} = \{\mathbf{h}_{(j_1, j_2)} \ (j_1, j_2) \in \{1, ..., l\} \times \{1, ..., m\}\}, \ |B_1| = n, \ |D_1| = lm,$$

$$(B \setminus B_1)/\theta = \{\{\beta^{(a,b)}_{\mathbf{h}_{(1,1)}}, ..., \beta^{(a,b)}_{\mathbf{h}_{(l,m)}}\} \mid a = \overline{1, l}, \ b = \overline{1, m}\} \text{ and}$$

$$z(\{\beta^{(a,b)}_{\mathbf{h}_{(1,1)}}, ..., \beta^{(a,b)}_{\mathbf{h}_{(l,m)}}\}) = \delta_{\mathbf{h}_{(a,b)}}.$$

Let $\mathbf{w} = (\delta_{\mathbf{h}_{(1,1)}}, ..., \delta_{\mathbf{h}_{(l,m)}})$ and $\mathbf{z} = (\beta^1, ..., \beta^n)$. We define

$$(1) \qquad h_{(j_1, j_2)}({}^t\mathbf{w}, {}^t\mathbf{z}) = \begin{cases} 1, \text{ if } s_{j_1} = f(s_{\mathbf{w}}, i_{\mathbf{z}}) \text{ and } o_2 = g(s_{\mathbf{w}}, i_{\mathbf{z}}) \\ 0, \text{ otherwise,} \end{cases}$$

where

$$s_{\mathbf{w}} = \begin{cases} s_{j_1'}, \text{ with } j_1 + j_2'' = \min\{j_1 + j_2 \mid {}^t\delta_{\mathbf{h}_{(j_1, j_2)}} = 1\} \\ s_1, \text{ if } {}^t\delta_{\mathbf{h}_{(j_1, j_2)}} = 0, \ \forall (j_1, j_2), \end{cases}$$

and

$$i_{\mathbf{z}} = \begin{cases} i_j, \text{ with } j = \min\{j_1 \mid {}^t\beta^{j_1} = 1\} \\ i_1, \text{ if } {}^t\beta^{j_1} = 0, \ \forall j_1. \end{cases}$$

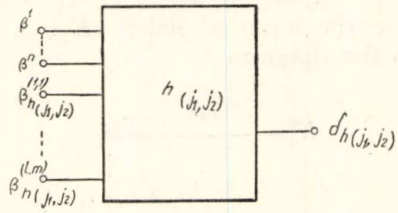

Fig. 2

The 2 - device $\mathbf{h}_{(j_1, j_2)}$ is represented in fig. 2. The automaton $A_{\mathbf{R}_A}$ constructed as in remark 2 simulates the automaton A. Indeed, the triple $\psi = (t, u, v)$, where the mappings $t : K \to I_2^n$, $u : S \to I_2^{lm}$, $v : I_2^{lm} \to O$ are given by $t(i_j) = \chi_{\{i_j\}} \in \mathcal{B}_2^n$, $u(s_j) = \chi_{\{\mathbf{h}_{(j,1)}\}} \in \mathcal{B}_2^{lm}$,

$$v(\mathbf{w}) = \begin{cases} o_{j_2}, \text{ where } j_1 + j_2 = \min(j_1' + j_2' \mid \delta_{\mathbf{h}(j_1', j_2')} = 1\}) \\ o_1, \text{ if } \delta_{\mathbf{h}(j_1', j_2')} = 0, \ \forall (j_1', j_2'), \end{cases}$$

makes commutative the diagram :

$$\begin{tikzcd} K^* \arrow[r, "g_{s_k}"] \arrow[d, "t"'] & O \\ (I_2^n)^* \arrow[r, "g_{\mathbf{R}_A} u(s_k)"] & I_2^{lm} \arrow[u, "v"'] \end{tikzcd}$$

Here χ_{M_1} is the characteristic vector of the subset M_1 of the set M and $\mathcal{B}_2 = \{0, 1\}$ is the two element Boolean algebra. Obviously, $\chi_{M_1} \in \mathcal{B}_2^{|M|}$

To prove the commutativity of the diagram comes to prove $g(s_k, i_j) = v(g_{\mathbf{R}_A}(u(s_k), t(i_j)))$.

But, $g_{\mathbf{R}_A}(\chi_{\{\mathbf{h}_{(k,1)}\}}, \chi_{\{i_j\}})$ can be computed taking into account that at the time t the input signals of the device $\mathbf{h}_{(j_1, j_2)}$ are presented in fig. 3 and the output of this device will be :

$$^{t+1}\delta_{\mathbf{h}_{(j_1, j_2)}} = h_{(j_1, j_2)}((0, \ldots, 1, \ldots, 0), (0, \ldots, 1, \ldots 0)) =$$

$$= \begin{cases} 1, \text{ if } s_{j_1} = f(s_k, i_j) \text{ and } o_{j_2} = g(s_k, i_j) \\ 0, \text{ otherwise,} \end{cases}$$

due to formula (1).

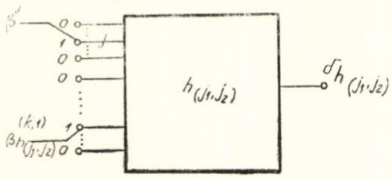

Fig. 3

We note that, at time $t + 1$ only the output of the device $\mathbf{h}_{(j_1, j_2)}$ is 1. Hence $v(g_{\mathbf{R}_A}(\chi_{\{\mathbf{h}_{(k,1)}}, \chi_{\{i_j\}})) = o_j = g(s_k, i_j)$ and the proof is completed.

Remark 3. Every modular network $\mathbf{R}$ can be simulated by a network of 2 — devices $\mathbf{R}_2$ (in the sense that the automaton $A_{\mathbf{R}}$ is simulated by the automaton $A_{\mathbf{R}_2}$).

The above consideration can be developed for non-finite alphabets.

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ASUPRA UNEI FORMALIZĂRI A REŢELELOR ABSTRACTE MODULARE

(Rezumat)

Se prezintă o descriere formală a noţiunii de reţea modulară introduă de McCullogh şi Pitts [3], ceea ce permite considerarea reţelelor modulare ca obiecte matematice.

Se studiază apoi posibilităţile reţelelor modulare de a simula automate finite.

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D. C. 511

AN APPLICATION OF RADO'S INEQUALITY

BY

P. S. BULLEN

1. Introduction

In several recent papers, (a fairly complete list can be found in [2]), various refinements of the arithmetic-geometric mean inequality have been given. Interesting though these results are in the general theory of inequalities they seem to have found no applications elsewhere unlike the original inequality which of course has found many uses. It would be of interest to consider the various places where the arithmetic-geometric mean inequality is applied to see if the various results can be improved by using one or other of the above mentioned refinements. This note makes on attempt at this for one such application namely in the proof of Hölder's inequality.

2. Notation and Basic Results

In this note all sequences are sequences of positive numbers. The weighted arithmetic and geometric means are

$$A_n(a\,;\,q) = \frac{1}{Q_n}\sum_{k=1}^{n} a_k\, q_k\,,$$

$$G_n(a\,;\,q) = \left(\prod_{k=1}^{n} a_k^{q_k}\right)^{\frac{1}{Q_n}},$$

respectively, where $Q_n = \sum_{k=1}^{n} q_k$. If $q_1 = \ldots = q_n$ we will write just A_n, and G_n, respectively.

The basic result connecting these two means is the famous arithmetic-geometric mean inequality:

$$A_n(a\,;\,q) \geqslant G_n(a\,;\,q), \tag{1}$$

with equality if and only if $a_1 = \ldots = a_n$.

The following refinement of (1) can be found in [1], [2]:

$$Q_n\,\{A_n(a\,;\,q) - G_n(a\,;\,q)\} \geqslant Q_{n-1}\,\{A_{n-1}(a\,;\,q) - G_{n-1}(a\,;\,q)\}, \tag{2}$$

with equality if and only if $a_n = G_{n-1}(a\,;\,q)$. Inequality (2) is a generalisation of R a d o ' s inequality, [3],

$$n(A_n - G_n) \geqslant (n-1)\,(A_{n-1} - G_{n-1}), \tag{2'}$$

equality occuring only if $a_n = G_{n-1}$.

Clearly repeated application of (2) gives (1); conversely inequality (1) implies (2), see [1] and also below. However it is clear that in general (2) is a stronger result than (1′) this apparent paradox is added to by the following argument that gives a proof that

$$A_n \geqslant G_n, \tag{1'}$$

with equality if and only if $a_1 = \ldots = a_n$.

(a) We first prove (1)′ when all but one of the $a_1, \ldots, a_n$ are equal. That is if $a > 0$, $b > 0$, n a positive integer then

$$\frac{1}{n}\,a + \frac{n-1}{n}\,b > a^{\frac{1}{n}}\,b^{\frac{n-1}{n}} \tag{3}$$

with equality if and only if $a = b$. This by putting $x = a/b$ is seen to be equivalent to the following; if n is a positive integer then for all $x > 0$

$$\{x + (n-1)\}^n \geqslant n^n x \tag{4}$$

with equality if and only if $x = 1$. But consider the polynomial $P(x) = \{x + (n-1)\}^n - n^n x$; by Descarte's Rule of Signs P has at most two positive roots; but $x = 1$ is a double root of P and $P(0) > 0$ and $P(2) > 0$. Hence for $x > 0$, $P(x) > 0$ unless $x = 1$ when $P(x) = 0$, which completes the proof of (3).

(b) Simple manipulations of (2′) show that it is equivalent to

$$\frac{1}{n}\,a_n + \frac{n-1}{n}\,G_{n-1} \geqslant a_n^{\frac{1}{n}}\,G_{n-1}^{\frac{n-1}{n}}$$

which is immediate from (a); equality holds if and only if $a_n = G_{n-1}$. This completes the proof of (2′) using the special case (3) of (1′).

(c) Finally by (2′),

$$n(A_n - G_n) \geqslant (n-1)\,(A_{n-1} - G_{n-1}) \geqslant \ldots \geqslant A_1 - G_1 = 0,$$

which gives $A_n \geqslant G_n$, with equality if and only if $a_1 = \ldots = a_n$, completing the proof of (1′).

3. Hölder's Inequality

In this section let $r_i > 0$, $i = 1, \ldots, n$ and put $\rho_n = \sum_{i=1}^{n} \frac{1}{r_i}$; for symmetry in our result it is convenient not to assume, as is usuallly done, that $\rho_n = 1$. The following result and proof are classic, [3] .

Theorem 1. *If $r_i > 0$, $i = 1, \ldots, n$ then*

$$H_n(b) = \frac{\sum_{j=1}^{m} \left(\prod_{i=1}^{n} b_{ij} \right)^{\frac{1}{\rho_n}}}{\prod_{i=1}^{n} \left(\sum_{j=1}^{m} b_{ij}^{r_i} \right)^{\frac{1}{r_i \rho_n}}} \leqslant 1 \tag{5}$$

with equality if and only if the sequences $\{b_{11}^{r_1}, \ldots, b_{1m}^{r_1}\}, \ldots, \{b_{n1}^{r_n}, \ldots, b_{nm}^{r_n}\}$ are proportional.

Proof. From (1) we have immediately that

$$\frac{1}{\rho_n} \sum_{i=1}^{n} \frac{a_i}{r_i} \geqslant \left(\prod a_i^{\frac{1}{r_i}} \right)^{\frac{1}{\rho_n}},$$

and inequality (5) follows by putting

$$a_i = \frac{b_{ij}^{r_i}}{\sum_{j=1}^{m} b_{ij}^{r_i}}, \quad j = 1, \ldots, m, \tag{6}$$

and summing over the index j. The case of equality is immediate.

Thus (5) is on immediate consequence of (1) and we obtain the following refinement of (5) by using (2).

Theorem 2. *If $r_i > 0$, $i = 1, \ldots, n$, then with the notation of Theorem 1*

$$\rho_n H_n(b) \leqslant \rho_{n-1} H_{n-1}(b) + \frac{1}{r_n} \tag{7}$$

with equality if and only if

$$\frac{b_{nj}}{\sum_{j=1}^{m} b_{nj}^{r_n}} = \left(\prod_{i=1}^{n-1} \frac{b_{ij}}{\left(\sum_{j=1}^{m} b_{ij}^{r_i} \right)^{1/r_i}} \right)^{\frac{1}{\rho_{n-1}}}, \quad j = 1, \ldots, m.$$

Proof. From (2) we have that

$$\rho_n\left\{\frac{1}{\rho_n}\sum_{i=1}^{n}\frac{a_i}{r_i}-\left(\prod_{i=1}^{n}a_i^{\frac{1}{r_i}}\right)^{\frac{1}{\rho_n}}\right\}\geqslant$$

(8)

$$\geqslant \rho_{n-1}\left\{\frac{1}{\rho_{n-1}}\sum_{i=1}^{n-1}\frac{a_i}{r_i}-\left(\prod_{i=1}^{n-1}a_i^{\frac{1}{r_i}}\right)^{\frac{1}{\rho_{n-1}}}\right\}.$$

Now making the identification (6) in (8) and summing over the index j we get that

$$\rho_n(1-H_n(b))\geqslant \rho_{n-1}(1-H_{n-1}(b))$$

which is immediately seen to be equivalent to (7). The case of equality follows in the usual way.

Remarks. (1) If $n=2$ then inequality (7) is just (5) for this value of n.

(2) Repeated application of (7) leads to (5), for we have that,

$$\rho_n H_n(b)\leqslant \rho_{n-1}H_{n-1}(b)+\frac{1}{r_n}\leqslant \rho_{n-2}H_{n-2}(b)+\frac{1}{r_{n-1}}+\frac{1}{r_n}\leqslant \ldots$$

$$\leqslant \rho_1 H_1(b)+\frac{1}{r_2}+\ldots+\frac{1}{r_n}=\rho,$$

which is just (5); the case of equality is also given by the argument.

(3) Inequality (2) can be read as saying that

$$Q_{n-1}(A_{n-1}(a;q)-G_{n-1}(a;q))=\inf_{a_n}\{Q_n(A_n(a;q)-G_n(a;q))\},$$

the inf being attained by choosing $a_n=G_{n-1}$. A similar interpretation can be given to (7).

(4) Alternatively since in (2) the order of the terms in $a_1,\ldots,a_n$ is immaterial, provided we associate q_i with a_i, no matter what is the actual position of a_i, $i=1,\ldots,n$, we can improve this inequality to,

$$Q_n(A_n(a;q)-G_n(a;q))\geqslant \sup_k Q_{n-1}^{(k)}\{A_{n-1}(a^{(k)};q^{(k)})-G_{n-1}(a^{(k)};q^{(k)})\}$$

where

$$Q_{n-1}^{(k)}=\sum_{\substack{j=1\\ j\neq k}} q_j\,,$$

$$a^{(k)}=(a_1,\ldots,a_{k-1},a_{k+1},\ldots,a_n)\,,\quad q^{(k)}=(q_1,\ldots,q_{k-1},q_{k+1},\ldots,q_n).$$

A similar refinement is possible for (7).

(5) A simpler form of the remark (4) which at the same time is more practical follows from the inequality

$$Q_n(A_n(a;q)-G_n(a;q))\geqslant Q_2(A_2(a;q)-G_2(a;q)).$$

By re-ordering both $a_1, \ldots, a_n$, and $q_1, \ldots, q_n$ this gives,

$$Q_n \{ A_n(a\,;\,q) - G_n(a\,;\,q) \} \geqslant$$

$$\geqslant \sup_{1 \leqslant i,\, \rho \leqslant n} \{ q_i\, a_i + q_j\, a_j - (q_i + q_j)\, (a_i^{a_i}\, a_j^{q_j})^{\frac{1}{q_i + q_j}} \}.$$

In the case of equal weights this is particularly simple, [2],

$$A_n - G_n \geqslant \frac{1}{n} \sup_{1 < i,\, j < n} \{ (\sqrt{a_i} - \sqrt{a_i})^2 \}.$$

Again this can be done for inequality (7) giving

$$\rho_n\, H_n(b) \leqslant \inf_{1 \leqslant i,\, m < n} \frac{r_i + r_k}{r_i r_k} \left\{ \frac{\sum_{j=1}^{m} (b_{ij}\, b_{kj})^{\frac{r_i r_k}{r_i + r_k}}}{\left(\sum_{j=1}^{m} b_{ij}^{r_i} \right)^{\frac{r_k}{r_i + r_k}} \left(\sum_{j=1}^{m} b_{kj}^{r_k} \right)^{\frac{r_i}{r_i + r_k}}} \right\}.$$

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O APLICAŢIE A INEGALITĂŢII LUI RADO

(Rezumat)

Autorul indică o interesantă aplicaţie a inegalităţii lui Rado în demonstraţia inegalităţii lui Hölder.

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MECANICĂ TEORETICĂ, FIZICĂ

C.Z. 513.83

SOME REMARKS CONCERNING THE MORSE THEORY OF THE COVERING SPACES OF MANIFOLDS[1])

BY

I. POP

In this paper it is proved that any smooth covering map having a finite number of leaves is homotopically equivalent with a covering map in the CW-complexes category.

Let $p: M \to N$ be a smooth covering map, M and N being smooth manifolds with boundary.

If $f: N \to R$ is a smooth function and $y = p(x)$ is a critically non-degenerate point of index q for f, then x is a critical non-degenerate point of index q for the map fp and reciprocally. It follows that if p is a covering map with n leaves, then to each critical non-degenerate point y, of index q, for f correspond n critical non-degenerate points for fp that have also index q. These are all the points of fibre $p^{-1}(y)$. It also follows that if M and N are two smooth manifolds and p has n leaves and f has h critical non-degenerate points, then fp has nh critical non-degenerate points.

Proposition 1. *If $f: N \to R$ is a smooth function, $a < b$, $p: M \to N$ is a smooth covering map with a finite number of leaves, $f^{-1}[a, b]$ is compact and does not contain any critical point for f, then there exists an isomorphism of the covering maps $p/M^a; M^a \to N^a$ and $p/M^b: M^b \to N^b$, where*

$$M^a = \{x \in M / fp(x) \leqslant a\}, \quad M^b = \{x \in M / fp(x) \leqslant b\}$$

and

$$N^a = \{y \in N / f(y) \leqslant a\}, \quad N^b = \{y \in N / f(y) \leqslant b\}.$$

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[1]) Presented at the scientific session of the University „Al. I. Cuza" Jassy, October 26—28, 1973.

P r o o f. To prove this, we shall show that there exist the diffeomorfisms $\varphi : M^a \to M^b$, $\psi : N^a \to N^b$, such that the diagram

$$\begin{array}{ccc} M^a & \xrightarrow{\quad p/M^a \quad} & N^a \\ \varphi \downarrow & & \downarrow \psi \\ M^b & \xrightarrow{\quad p/M^b \quad} & N^b \end{array}$$

be commutative.

Let $\rho : N \to R$ be a smooth function defined as follows:

$$\rho(y) = \frac{1}{< \operatorname{grad} f, \operatorname{grad} f >_y} \text{ on compact set } f^{-1}[a, b] \text{ and } \rho(y) = 0 \text{ without}$$

some compact neighbourhood V of the set $f^{-1}[a, b]$. Because p has a finite number of leaves, there follows that p is a proper map and then $(fp)^{-1}[a, b]$ is compact. We define the map $\rho' : M \to R$ as folows:

$$\rho'(x) = \frac{1}{< \operatorname{grad} fp, \operatorname{grad} fp >_x} \text{ on the set } (fp)^{-1}[a, b] \text{ and } \rho'(x) = 0$$

for $x \in p^{-1}(V)$ that is a compact neighbourhood of the set $(fp)^{-1}[a, b]$. Because p is a smooth covering map, p is a local diffeomorphism and then we can suppose that it is an isometry. Then it easily comes out that $p_*(\operatorname{grad} fp)_x = (\operatorname{grad} f)_{p(x)}$, which implies $\rho'(x) = \rho(p(x))$, for every $x \in M$.

Now, we consider the vectors' fields Y over N and X over M by the formulas $Y_y = \rho(y)(\operatorname{grad} f)_y$, for $y \in N$ and $X_x = \rho'(x)(\operatorname{grad} fp)_x$, for $x \in M$. From the above relation we obtain $p_* X_x = Y_{p(x)}$, for every $x \in M$. Then, the uniparametrical groups ψ_t and φ_t defined by Y and respectively X, verify the relation $p\varphi_t = \psi_t p$. In particular, the relation $p\varphi_{b-a} = \psi_{a-b} p$. It is known [1], that $\psi = \psi_{b-a}/M^a$ applies diffeomorfically M^a on M^b and $\varphi = \varphi_{b-a}/N^a$ applies diffeomorfically N^a on N^b.

P r o p o s i t i o n 2. *Let $p : M \to N$ be a smooth covering map with n leaves, $f : N \to R$ a smooth function having $y_0 \in N$ a critical non-degenerate point of index q, the set $f^{-1}[c - \varepsilon, c + \varepsilon]$ being compact and containing no other critical points for f without y_0, where $c = f(y_0)$.*

Then, for $\varepsilon > 0$ sufficiently small, there exist the attaching maps $e^q \supset \dot{e}^q \xrightarrow{g} N^{c-\varepsilon}$ and $e_i^q \supset \dot{e}_i^q \xrightarrow{g_i} M^{c-\varepsilon}$, $i = 1, \ldots, n$, $e_i^q = e^q$ and a covering map $p' : M^{c-\varepsilon} \cup_{(g_i)} e_i^q \to N^{c-\varepsilon} \cup_g e^q$ whose restriction to $N^{c-\varepsilon}$ is $p/M^{c-\varepsilon}$, and besides, p' is homotopically equivalent with the covering map $p/M^{c+\varepsilon} : M^{c+\varepsilon} \to N^{c+\varepsilon}$.

P r o o f. By assumption it results, [1], that there exists the attaching map g such that $N^{c+\varepsilon}$ is homotopically equivalent with the space $N^{c-\varepsilon} \cup_g e^q$.

On the other hand, because $(fp)^{-1}(c) = p^{-1}(y_0)$ contains n critical points of index q for the map fp, there exist the attaching maps g_i, $i = 1, \ldots, n$ such that the spaces $M^{c+\varepsilon}$ and $M^{c-\varepsilon} \cup_{(g_i)} e_i^q$ have the same homotopy type. From Proposition 1 and from the construction of the attaching maps g and g_i[1], it comes out that we have the following commutative diagram

$$\begin{tikzcd} M^{c-\varepsilon} \arrow[rr, "p/M^{c-\varepsilon}"] & & N^{c-\varepsilon} \\ & e^q \arrow[ul, "g_i"] \arrow[ur, "g"'] & \end{tikzcd}$$

for every $i = 1, \ldots, n$.

If we consider the trivial covering map $\vee e_i^q \to e^q$ (identified with $e^q \times \{1, \ldots, n\} \to e^q$), and we sum with $p/M^{c-\varepsilon}$ we obtain a covering map $P : M^{c-\varepsilon} \vee (\vee e_i^q) \to N^{c-\varepsilon} \vee e^q$. By considering the relations $pg_i = g$ we obtain a map

$$p' : M^{c-\varepsilon} \cup_{(g_i)} e_i^q \to N^{c-\varepsilon} \cup_g e^q$$

so that there exists the following commutative diagram :

$$\begin{tikzcd} M^{c-\varepsilon} \cup_{(g_i)} e_i^q \arrow[r, "p'"] & N^{c-} \cup_g e^q \\ M^{c-\varepsilon} \vee (\vee e_i^q) \arrow[u, "\pi"] \arrow[r, "P"] & N^{c-\varepsilon} \vee e^q \arrow[u, "\pi'"'] \end{tikzcd}$$

where π and π' are the canonical projections. Because the maps g and g_i are injective, it results that π and π' are local homeomorphisms, which implies that p' is a covering map. We have

$$p'/M^{c-\varepsilon} = P/M^{c-\varepsilon} = p/M^{c-\varepsilon}.$$

Now, let $F : N \to R$ function be a smooth function having the properties

a) $F^{-1}(-\infty, c + \varepsilon] = N^{c+\varepsilon}$,

b) F has the same critical points as f,

c) $F^{-1}(-\infty, c - \varepsilon)$ is a retract by deformation of $M^{c+\varepsilon}$ [1].

We note $H_f = \overline{F^{-1}(-\infty, c - \varepsilon] - N^{c-\varepsilon}}$. We consider the map Fp that will have analogous properties with respect to fp. From Proposition 1, we have the isomorphism of the covering maps :

$$p/M^{c-\varepsilon} \cup H_{fp} : M^{c-\varepsilon} \cup H_{fp} \to N^{\varepsilon-c} \cup H_f$$

and

$$p/M^{c+\varepsilon} : M^{c+\varepsilon} \to N^{c+\varepsilon}.$$

There also exists the retracts by deformation:

$$r_1 : N^{c-\varepsilon} \cup H_f \to N^{c-\varepsilon} \cup_g e^q \quad \text{and}$$

$$r_1' : M^{c-\varepsilon} \cup H_{fp} \to M^{c-\varepsilon} \cup_{(gi)} e_i^q,$$

so that there exists the following commutative diagram:

$$\begin{tikzcd}
M^{c+\varepsilon} \arrow[r, "p/M^{c+\varepsilon}"] \arrow[d, "\varphi"'] & N^{c+\varepsilon} \arrow[d, "\psi"] \\
M^{c-\varepsilon} \cup H_{fp} \arrow[r, "p/M^{c-\varepsilon} \cup H_{fp}"] \arrow[d, "r_1'"'] & N^{c-\varepsilon} \cup H_f \arrow[d, "r_1"] \\
M^{c-} \cup_{(g_i)} e_i^q \arrow[r, "p'"] & N^{c-\varepsilon} \cup_g e^q
\end{tikzcd}$$

where φ and ψ are diffeomorphisms. This proves the proposition.

From Propositions 1 and 2 it results the following

Theorem. *Let $p : M \to N$ be a smooth covering map with a finite number of leaves and $f : N \to R$ a smooth function without any critical degenerate point, so that for every $a \in R$ the manifold N^a is compact. In these conditions, there exist a covering map $p' : M' \to N'$ in the CW-complexes category which is homotopically equivalent with p.*

Corollary. Every smooth covering map $p : M \to N$ with a finite number of leaves is homotopically equivalent with a covering map in the *CW*-complexes category.

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UNELE OBSERVAŢII PRIVIND TEORIA MORSE A SPAŢIILOR DE ACOPERIRE ALE VARIETĂŢILOR

(Rezumat)

În această Notă se demonstrează că orice proiecţie de acoperire netedă avînd un număr finit de foi este echivalentă homotopic cu o proiecţie de acoperire în categoria CW-complexelor.

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Secţia I
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C.Z. 512.9

STEINITZ'S DUAL THEOREM

BY

GH. COSTOVICI

The central role of Steinitz's theorem in the theory of the basis in a finite dimensional vector space is known. Here, we present „Steinitz's dual theorem" on which we shall furnish evidence of the same role.

If V is a vector space over a division ring K, we recall that $B \subset V$ is an independent set $\Leftrightarrow \forall\, \beta_1 b_1 + \cdots + \beta_s b_s = 0,\ b_i \in B,\ \beta_i \in K,\ i = \overline{1, s} \Rightarrow \beta_i = 0,\ i = \overline{1, s}$, and that $G \subset V$ is a generator set $\Leftrightarrow \forall\, v \in V$, $\exists g_i \in G$ and $\alpha_i \in K$, $i = \overline{1, r}$, so that $v = \alpha_1 g_1 + \alpha_2 g_2 + \cdots + \alpha_r g_r$.

Steinitz's dual theorem. *If in vector space V over a division ring K, $\{u_1, u_2, \cdots, u_n\}$ is a generator set and $\{v_1, v_2, \cdots, v_r\}$ is an independent set, then $r \leqslant n$ and, by a possible renumbering, $\{v_1, \cdots, v_r, u_{r+1}, \cdots, \cdots, u_n\}$ is a generator set.*

This theorem will be proved by induction on r. If $r = 1$, obvious $r \leqslant n$. $\{u_1, \cdots, u_n\}$ being a generator set $\Rightarrow v_1 = \alpha_1 u_1 + \cdots + \alpha_n u_n$ for some $\alpha_i \in K$, $i = \overline{1, n}$, and then, evidently, $\{v_1, u_2, \cdots, u_n\}$ is a generator set. We assume $r > 1$ and that the theorem is valid for $r - 1$. $\{v_1, v_2, \cdots, v_r\}$ being independent $\Rightarrow \{v_1, v_2, \ldots, v_{r-1}\}$ is independent $\Rightarrow$ (by hypothesis), $r - 1 \leqslant n$ and that $\{v_1, \cdots, v_{r-1}, u_r, \cdots, u_n\}$ is a generator set. If $r - 1 = n$, then $\{v_1, v_2, \cdots, v_{r-1}\}$ is a generator set and therefore $v_r = \beta_1 v_1 + \cdots + \beta_{r-1} v_{r-1}$ for some $\beta_i \in K$, $i = \overline{1, r-1}$ and $\{v_1, v_2, \cdots, v_r\}$ should be not an independent set. Hence, $r - 1 < n$ and therefore $r \leqslant n$. $\{v_1, v_2, \cdots, v_{r-1}, u_r, u_{r+1}, \cdots, u_n\}$ being a generator set, we have $v_r = \gamma_1 v_1 + \cdots + \gamma_{r-1} v_{r-1} + \gamma_r u_r + \gamma_{r+1} u_{r+1} + \cdots + \gamma_n u_n$ for some $\gamma_i \in K$, $i = \overline{1, n}$. If all γ_i were zero, $i = \overline{r, n}$, then $\{v_1, v_2, \cdots, v_r\}$ should be not an independent set. Hence, $\exists \gamma_r \neq 0$ (by renumbering) and then $u_r = -\gamma_r^{-1}\gamma_1 v_1 - \gamma_r^{-1}\gamma_2 v_2 - \cdots - \gamma_r^{-1}\gamma_{r-1} v_{r-1} + \gamma_r^{-1} v_r - \gamma_r^{-1}\gamma_{r+1} u_{r+1} - \cdots - \gamma_r^{-1}\gamma_n u_n$ and therefore $\{v_1, v_2, \ldots, v_r, u_{r+1}, \ldots, u_n\}$ is a generator set.

Furthermore, we know that a subset E from a vector space V over a division ring K is a basis for $V \Leftrightarrow E$ satisfies one of the following equivalent conditions:

(a) E is a generator and independent set.

(b) E is a maximal independent set.

(c) E is a minimal generator set.

(d) E is a generator set and every vector from V has a unique representation.

It is not difficult to see how from the foregoing, one gets, the

Theorem. *If a vector space V over a division ring K has a basis with n vectors, then:*

(I) *every independent set contains at most n vectors;*

(II) *every independent set with n vectors is a basis;*

(III) *every generator set contains at least n vectors;*

(IV) *every generator set with n vectors is a basis;*

(V) *every basis has exactly n vectors.*

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DUALA TEOREMEI LUI STEINITZ

(Rezumat)

Se enunţă şi se demonstrează „duala teoremei lui Steinitz"; dacă $\{u_1, u_2, \ldots, u_n\}$ este o mulţime generatoare pentru un spaţiu vectorial V peste un corp K şi dacă $\{v_1, v_2, \ldots, v_r\}$ este o mulţime independentă din V, atunci $r \leqslant n$ şi, printr-o eventuală renumerotare, $\{v_1, v_2, \ldots, v_r, u_{r+1}, \ldots, u_n\}$ este o mulţime generatoare — teoremă din care rezultă o teorie a bazei într-un spaţiu vectorial finit dimensional.

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ASUPRA UNOR FAMILII DE ALGEBRE DE ORDIN TREI

DE
I. ENESCU

1. În [1] se prezintă cinetica de reacţie între trei specii chimice şi se semnalează reacţiile ireversibile cărora li se asociază algebrele chimice ce corespund, în sensul din [2], cineticii de reacţie. În cele ce urmează, ne propunem studiul algebrelor chimice de ordin trei, ce corespund unor reacţii consecutive cu o reacţie reversibilă. Cinetica unor asemenea reacţii, este descrisă de sistemul diferenţial:

$$
(1)\qquad \begin{cases} \dot{x}^1 = -2(x^1)^m (x^3)^{2-m}, \\ \dot{x}^2 = 2\mu(x^3 - x^1 - x^2)^n (x^3)^{2-n} - 2\nu(x^2)^q (x^3)^{2-q}, \\ \dot{x}^3 = 0, \end{cases}
$$

unde ordinele de reacţie m, n, $q = 0, 1, 2$ iar μ şi ν sînt parametri reali strict pozitivi [1].

Vom studia algebrele asociate sistemului (1), în cazurile $m = n = q = 0, 1, 2$, adică, respectiv, familiile de algebre de ordin trei reprezentate prin tabelele de înmulţire:

(2)

	u_1	u_2	u_3
u_1	0	0	0
u_2	0	0	0
u_3	0	0	$-2u_1+2(\mu-\nu)u_2$

,

(3)

	u_1	u_2	u_3
u_1	0	0	$-2u_1-2\mu u_2$
u_2	0	0	$(-2\mu-2\nu)u_2$
u_3	$-2u_1-2\mu u_2$	$(-2\mu-2\nu)\,u_2$	$2\mu u_2$

,

(4)

	u_1	u_2	u_3
u_1	$-2u_1+2\mu u_2$	$4\mu u_2$	$-2\mu u_2$
u_2	$4\mu u_2$	$(2\mu-2\nu)u_2$	$-4\mu u_2$.
u_3	$-2\mu u_2$	$-4\mu u_2$	$2\mu u_2$

2. Fie algebrele (2) şi să notăm $\mu-\nu=\alpha$, evident $\alpha\in R$. Atunci, făcînd schimbarea de bază

(5) $$u_1=e_1,\quad \alpha u_2=e_2,\quad u_3=e_3,\quad \alpha\neq 0,$$

se obţine tabela

(6)

	e_1	e_2	e_3
e_1	0	0	0 ,
e_2	0	0	0
e_3	0	0	$-2e_1+2e_2$

care arată că, toate algebrele (2) pentru care $\mu\neq\nu$, sînt izomorfe. În cazul $\alpha=0$, $(\mu=\nu)$ tabela (2) devine :

(7)

	u_1	u_2	u_3
u_1	0	0	0
u_2	0	0	0
u_3	0	0	$-2u_1$.

În sfîrşit, schimbînd din nou baza în (6), după

$$e_1-e_2=v_1,\quad e_1+e_2=v_2,\quad e_3=v_3,$$

se obţine tabela (7), ceea ce stabileşte

T e o r e m a 1. *Toate algebrele familiei* (2) *sînt izomorfe. Ele se pot reprezenta, de exemplu, prin aceea corespunzătoare cazului* $\mu=\nu$ *dat prin tabela* (7).

Facem observaţia, verificabilă prin calcul direct, că toate algebrele (2) sînt asociative.

3. În cazul familiei (3), să schimbăm baza, după

$$-2u_1-2\mu u_2=e_1,\quad 2\mu u_2=e_2,\quad u_3=e_3.$$

Se obţine tabela de înmulţire

(8)

	e_1	e_2	e_3
e_1	0	0	$-2e_1+2(\mu+\nu)e_2$
e_2	0	0	$-2(\mu+\nu)e_2$, ,
e_3	$-2e_1+2(\mu+\nu)e_2$	$-2(\mu+\nu)e_2$	e_2

care arată că *familia* (3) *depinde numai de suma* $\mu+\nu$.

O nouă schimbare de bază, efectuată în (8), anume:

$$-e_1 + \alpha e_2 = v_1, \qquad \alpha^2 e_2 = v_2, \qquad -\frac{1}{8\alpha} e_2 - \frac{1}{2} e_3 = v_3,$$

conduce la tabela

(8′)

	v_1	v_2	v_3
v_1	0	0	$v_1 + v_2$
v_2	0	0	αv_2,
v_3	$v_1 + v_2$	αv_2	0

unde $\alpha = \mu + \nu$.

Rezolvînd acum ecuaţia $x^3 = x^2$, se găsesc soluţiile:

$$(9) \quad \begin{cases} x_1 = b(1-\alpha)v_1 + bv_2 + v_3, & \forall b \in R \\ x_2 = bv_2 + \dfrac{1}{\alpha} v_3, & \forall b \in R \\ x_3 = av_1 + bv_2 & \forall a, b \in R. \end{cases}$$

Se constată imediat că, pentru orice valoare $\alpha \neq 1$, ecuaţia $x^3 = x^2$ are, în algebra corespunzătoare, trei soluţii şi acestea sînt date de (9). În cazul $\alpha = 1$, soluţiile x_1 şi x_2 coincid, ceea ce stabileşte

T e o r e m a 2. *Algebra corespunzătoare valorii* $\alpha = 1$ *a parametrului nu este izomorfă cu nici o alta din familia* (3).

Această algebră are tabela de înmulţire:

(10)

	v_1	v_2	v_3
v_1	0	0	$v_1 + v_2$
v_2	0	0	v_2
v_3	$v_1 + v_2$	v_2	0

În continuare, considerăm $\alpha \neq 1$ şi, în această ipoteză, să schimbăm baza în (8′) după relaţiile:

$$\frac{\alpha}{1-\alpha} v_1 - \frac{\alpha}{(1-\alpha)^2} v_2 = w_2, \qquad v_1 + \frac{1}{1-\alpha} v_2 = w_2, \qquad \frac{1}{\alpha} v_3 = w_3.$$

Obţinem tabela:

$$
(11) \qquad \begin{array}{c|ccc}
 & w_1 & w_2 & w_3 \\
\hline
w_1 & 0 & 0 & w_1 + w_2 \\
w_2 & 0 & 0 & \frac{1}{\alpha} w_2 \\
w_3 & w_1 + w_2 & \frac{1}{\alpha} w_2 & 0
\end{array} .
$$

ceea ce demonstrează

T e o r e m a **3**. *În familia* (3), *toate algebrele corespunzătoare valorilor* α *sînt izomorfe cu acelea ce corespund valorilor* $\frac{1}{\alpha}$ *ale parametrului.*

În cele ce urmează, considerăm $\alpha \in (0, 1)$.

Fie α_1 şi α_2 două valori ale parametrului, conţinute în (0, 1) şi, fie φ un operator care reprezintă izomorf algebra corespunzătoare valorii α_1, pe aceea ce corespunde valorii α_2. Să punem:

$$\varphi(v_1) = Av_1 + Bv_2 + Cv_3, \ \varphi(v_2) = Dv_1 + Ev_2 + Fv_3, \ \varphi(v_3) = Gv_1 + Hv_2 + Kv_3.$$

Condiţiile $\varphi^2(v_1) = \varphi^2(v_2) = \varphi^2(v_3) = \varphi(v_1)\,\varphi(v_2) = 0$ şi $\varphi(v_2)\,\varphi(v_3) = \alpha_2\,\varphi(v_2)$, conduc la

$$A = \frac{D}{\alpha_2 - 1}, \quad B = \frac{D}{(\alpha_1 - 1)(\alpha_2 - 1)}, \quad E = \frac{D}{1 - \alpha_1}, \quad K = \alpha_2,$$

unde D este arbitrar, iar $C = F = G = H = 0$. Atunci:

$$
\begin{cases}
\varphi(v_1) = \dfrac{D}{\alpha_2 - 1} v_1 + \dfrac{D}{(\alpha_1 - 1)(\alpha_2 - 1)} v_2, \\[2ex]
\varphi(v_2) = Dv_1 + \dfrac{D}{1 - \alpha_1} v_2, \\[2ex]
\varphi(v_3) = \alpha_2 v_3.
\end{cases}
$$

Condiţia $\varphi(v_1)\,\varphi(v_3) = \varphi(v_1) + \varphi(v_2)$, implică, cu necesitate, ca $2\alpha_1\alpha_2 - \alpha_2 = 2 - \alpha_2$, adică $\alpha_1\,\alpha_2 = 1$ ceea ce este imposibil, întrucît α_1 şi α_2 sînt, respectiv, subunitari. Am dovedit deci

T e o r e m a 4. *Toate algebrele* (3), *corespunzătoare valorilor subunitare ale parametrului* α, *sînt neizomorfe.*

Teoremele **3** şi 4 atrag imediat, următorul

C o r o l a r. *Există o infinitate de tipuri neizomorfe de algebre* (**3**), *corespunzătoare valorilor* $\alpha \in (0, 1)$. *Tipul* $\alpha = 1$ *este neizomorf cu oricare altul şi el este singurul caz reprezentat printr-o singură algebră* (**3**). *Toate*

celelalte se grupează în perechi după, respectiv, α *şi* $\frac{1}{\alpha}$, *fiecare asemenea pereche reprezentînd cîte un tip neizomorf de algebră* (**3**).

Observăm că algebrele (**3**) nu sînt asociative pentru nici o valoare a parametrului α.

4. Să cercetăm acum familia (4). Începem prin a-i afla idempotenţii şi, atunci, ecuaţia $x^2 = x$, conduce, în ipoteza $x \neq 0$, la soluţiile:

$$(12)\quad \left|\begin{aligned} E_1 &= -\frac{1}{2}u_1 + \frac{4\mu + 1 + \sqrt{12\mu^2 + 4\mu\nu + 8\mu}}{4(\mu - \nu)}u_2, \\ E_2 &= -\frac{1}{2}u_1 + \frac{4\mu + 1 - \sqrt{12\mu^2 + 4\mu\nu + 8\mu}}{4(\mu - \nu)}u_2, \\ E_3 &= \frac{1}{2(\mu - \nu)}u_2, \end{aligned}\right.$$

care arată că, în cazul $\mu \neq \nu$, algebrele (4) au trei idempotenţi distincţi daţi de relaţiile (12). În cazul $\mu = \nu$, ecuaţia $x^2 = x$ are o singură soluţie nebanală, anume

$$(13)\qquad E_0 = -\frac{1}{2}u_1 + \frac{\mu}{2(4\mu + 1)}u_2,$$

ceea ce stabileşte

T e o r e m a 5. *Toate algebrele* (4) *care corespund valorilor* $\mu = \nu$ *ale celor doi parametri, sînt neizomorfe cu, respectiv, oricare algebră* (4), *corespunzătoare valorilor* $\mu \neq \nu$ *ale parametrilor.*

Să cercetăm cazul $\mu \neq \nu$. Cu schimbarea de bază

$$(14)\qquad u_1 = \varepsilon_1, \quad 2\mu u_2 = \varepsilon_2, \quad u_1 + u_3 = \varepsilon_3,$$

se obţine, ţinînd seama de (4), tabela

(15)

	ε_1	ε_2	ε_3
ε_1	$-2\varepsilon_1 + \varepsilon_2$	$4\mu\varepsilon_2$	$-2\varepsilon_1$
ε_2	$4\mu\varepsilon_2$	$4\mu(\mu - \nu)\varepsilon_2$	0
ε_3	$-2\varepsilon_1$	0	$-2\varepsilon_1$

Ecuaţia $x^2 = 0$ are soluţiile nebanale, date de

$$(16)\quad \left\{\begin{aligned} x_1 &= \varepsilon_1 + \frac{-2\mu + \sqrt{3\mu^2 + \mu\nu}}{2\mu(\mu - \nu)}\varepsilon_2 - \varepsilon_3, \\ x_2 &= \varepsilon_1 + \frac{-2\mu - \sqrt{3\mu^2 + \mu\nu}}{2\mu(\mu - \nu)}\varepsilon_2 - \varepsilon_3. \end{aligned}\right.$$

Atunci, luînd ca bază ansamblul format din cei doi nilpotenţi (16) şi idempotentul E_3 din (12) care, în baza (14) are forma

$$E_3 = \frac{1}{4\mu(\mu - \nu)} \varepsilon_2,$$

să punem:

$$(17) \quad \begin{cases} v_1 = \varepsilon_1 + \dfrac{-2\mu + \sqrt{3\mu^2 + \mu\nu}}{2\mu(\mu - \nu)} \varepsilon_2 - \varepsilon_3, \\ v_2 = \varepsilon_1 + \dfrac{-2\mu - \sqrt{3\mu^2 + \mu\nu}}{2\mu(\mu - \nu)} \varepsilon_2 - \varepsilon_3, \\ v_3 = \dfrac{1}{4\mu(\mu - \nu)} \varepsilon_2. \end{cases}$$

În baza (17) obţinem tabela de înmulţire

(18)

	v_1	v_2	v_3
v_1	0	$-8\mu(3\mu+\nu)v_3$	$2\sqrt{3\mu^2+\mu\nu}\,v_3$
v_2	$-8\mu(3\mu+\nu)v_3$	0	$-2\sqrt{3\mu^2+\mu\nu}\,v_3$
v_3	$2\sqrt{3\mu^2+\mu\nu}\,v_3$	$-2\sqrt{3\mu^2+\mu\nu}\,v_3$	v_3

În sfîrşit, o ultimă schimbare de bază, efectuată în (18), anume:

$$w_1 = \frac{1}{2\sqrt{3\mu^2 + \mu\nu}} v_1, \quad w_2 = \frac{1}{-2\sqrt{3\mu^2 + \mu\nu}} v_2, \quad w_3 = v_3,$$

conduce la

(19)

	w_1	w_2	w_3
w_1	0	$2w_3$	w_3
w_2	$2w_3$	0	w_3
w_3	w_3	w_3	w_3

fapt care demonstrează

T e o r e m a 6. *În cazul* $\mu \neq \nu$, *toate algebrele* (4) *sînt izomorfe şi ele se pot reprezenta prin algebra al cărui produs este dat de* (19).

5. Rămîn de studiat algebrele (4) ce corespund la respectiv valori egale ($\mu = \nu$) ale parametrilor, adică algebrele date prin tabela

$$
(20) \qquad \begin{array}{c|ccc}
 & u_1 & u_2 & u_3 \\
\hline
u_1 & -2u_1 + 2\mu u_2 & 4\mu u_2 & -2\mu u_2 \\
u_2 & 4\mu u_2 & 0 & -4\mu u_2 \\
u_3 & -2\mu u_2 & -4\mu u_2 & 2\mu u_2
\end{array},
$$

Schimbînd baza după : $u_1 = \varepsilon_1$, $2\mu u_2 = \varepsilon_2$, $u_1 + u_3 = \varepsilon_3$, se obţine

$$
(20') \qquad \begin{array}{c|ccc}
 & \varepsilon_1 & \varepsilon_2 & \varepsilon_3 \\
\hline
\varepsilon_1 & -2\varepsilon_1 + \varepsilon_2 & 4\mu\varepsilon_2 & -2\varepsilon_1 \\
\varepsilon_2 & 4\mu\varepsilon_2 & 0 & 0 \\
\varepsilon_3 & -2\varepsilon_1 & 0 & -2\varepsilon_1
\end{array},
$$

Căutînd nilpotenţii de indice doi, se găseşte că, soluţiile nebanale ale ecuaţiei $x^2 = 0$, pentru algebra (20), sînt

$$
(21) \qquad x_1 = -8\mu\varepsilon_1 + \varepsilon_2 + 8\mu\varepsilon_3, \qquad x_2 = \varepsilon_2.
$$

Pe de altă parte, există un singur idempotent propriu, care, în baza (20′), are expresia

$$
(13') \qquad E_0 = -\frac{1}{2}\varepsilon_1 + \frac{1}{4(4\mu + 1)}\varepsilon_2 .
$$

Să schimbăm din nou baza, după :

$$
v_1 = \frac{1}{2\mu^2}x_1, \qquad v_2 = -\frac{1}{2\mu(4\mu + 1)}x_2, \qquad v_3 = E_0.
$$

Se obţine tabela

$$
(22) \qquad \begin{array}{c|ccc}
 & v_1 & v_2 & v_3 \\
\hline
v_1 & 0 & 16\,v_2 & -2v_2 \\
v_2 & 16\,v_2 & 0 & -2\mu v_2 \\
v_3 & -2v_2 & -2\mu v_2 & v_3
\end{array}.
$$

Fie acum μ_1 şi μ_2 două valori arbitrare date parametrului μ şi, se notăm prin φ o funcţie care reprezintă izomorf algebra corespunzătoară valorii μ_1 a parametrului pe aceea ce corespunde valorii μ_2. Avem :

$$
(23) \qquad \begin{gathered}
\varphi(v_1) = Av_1 + Bv_2 + Cv_3, \qquad \varphi(v_2) = Dv_1 + Ev_2 + Fv_3, \\
\varphi(v_3) = Lv_1 + Mv_2 + Nv_3.
\end{gathered}
$$

Faptul că operatorul φ reprezintă izomorf cele două algebre, implică a fortiori, ca $\varphi^2(v_1) = 0$, $\varphi^2(v_2) = 0$, $\varphi^2(v_3) = v_3$, ceea ce conduce la :

$\varphi(v_1) = Bv_2$ sau $\varphi(v_1) = Av_1$, A, B arbitrari nenuli ;
$\varphi(v_2) = Ev_2$ sau $\varphi(v_2) = Dv_1$, E, D, arbitrari nenuli ; $\varphi(v_3) = v_3$.

Condiţia $\varphi(v_1)\,\varphi(v_2) = 16\,\varphi(v_2)$ implică în mod necesar, $\varphi(v_1) = v_1$, iar condiţia $\varphi(v_1)\,\varphi(v_3) = -2\varphi(v_2)$ arată că $\varphi(v_2) = v_2$. Se verifică aceste rezultate, constatînd că relaţia $\varphi(v_2)\,\varphi(v_3) = -2\mu_2\,\varphi(v_2)$ nu poate avea loc decît pentru $\mu_1 = \mu_2$, cu alte cuvinte, singurul izomorfism posibil între două algebre arbitrare din familia (20), este cel identic. Am stabilit astfel

Teorema 7. *Toate algebrele* (4) *ce corespund la valori egale ale celor doi parametri sînt neizomorfe.*

6. În încheiere, observăm că familia (2) nu conţine algebre izomorfe cu respectiv nici o algebră din familiile (3) şi (4), întrucît toate algebrele (2) sînt asociative celelalte neavînd această proprietate. La fel, nici familia (3) nu conţine algebre izomorfe cu (4), pentru că nici o algebră (3) nu are idempotenţi proprii.

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SUR CERTAINES FAMILLES D'ALGÈBRES D'ORDRE TROIS

(Résumé)

En considérant les réactions chimiques consécutives avec une réaction réversible entre trois espèces chimiques, dont la cinétique est donnée par le système (1), on étudie les familles d'algèbres d'ordre trois associées (2)–(4), dans les cas $m = n = q = 0, 1, 2$.

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O OBSERVAŢIE ASUPRA TEOREMELOR LUI SCHAUDER

DE

AL. SABADÎŞ şi GH. FLOREA

În cele ce urmează vom nota cu (X, τ) un spaţiu local convex, a cărui topologie este generată de o familie suficientă de seminorme $\{s_i | i \in I\}$.

Amintim cîteva noţiuni.

D e f i n i ţ i a 1. Spunem că un spaţiu are proprietatea punctului fix dacă orice aplicaţie continuă, definită pe el, cu valori în el însuşi, admite cel puţin un punct fix. Evident, un spaţiu cu această proprietate este, cu necesitate, topologic.

D e f i n i ţ i a 2. Fie $A \subset (X, \tau)$, τ fiind topologia generată de familia suficientă de seminorme $\{s_i | i \in I\}$. Spunem că mulţimea A are proprietatea punctului fix, dacă (A, τ_A) este un spaţiu cu această proprietate.

D e f i n i ţ i a 3. Fie spaţiul şirurilor de pătrat sumabile, l_2. Mulţimea şirurilor $\{x_n\} \in l_2$ cu proprietatea $|x_n| \leqslant \frac{1}{n}$, $\forall n \in N$, se numeşte cubul lui Hilbert.

Notăm cu H această mulţime şi arătăm că este convexă şi compactă. Fie $x_n \in H$ şi $y_n \in H$. Avem:

$$|\alpha x_n + \beta y_n| \leqslant |\alpha x_n| + |\beta y_n| = |\alpha|\,|x_n| + |\beta|\,|y_n| \leqslant |\alpha| \frac{1}{n} + |\beta| \frac{1}{n} =$$
$$= \frac{1}{n}(|\alpha| + |\beta|) = \frac{1}{n}.$$

Presupunem că H este compactă şi nu este total mărginită. Fie $H^1 = \{x_1\}$ o mulţime cu proprietatea că $\forall \varepsilon > 0$, există $x_2 \in H$ astfel ca $\left[\sum_{i=1}^{\infty} (x_1^i - x_2^i)^2\right]^{1/2} \geqslant$ $\geqslant \varepsilon$. Mulţimea $H^2 = \{x_1, x_2\}$ satisface aceeaşi proprietate pentru că, con

trar, ea ar fi o ε - reţea finită. Există deci, un $x_3 \in H$, cu $\left[\sum_{i=1}^{\infty}(x_3^i - x_1^i)^2\right]^{1/2} \geqslant \varepsilon$ şi $\left[\sum_{i=1}^{\infty}(x_3^i - x_2^i)^2\right]^{1/2} \geqslant \varepsilon$. Continuînd acest raţionament vom obţine un şir $\{x_n\}$ astfel încît $\left[\sum_{i=1}^{\infty}(x_n^i - x_m^i)^2\right]^{1/2} \geqslant \varepsilon$, $\forall m,\ n \in N,\ m \neq n$. Rezultă că şirul $\{x_n\}$ nu are subşiruri convergente ceea ce contrazice compacitatea lui H. Să demonstrăm acum că orice şir $\{x_n\} \in H$ admite un subşir convergent. Luăm un şir numeric $\varepsilon_n \to 0$ şi fie $H^1_{\varepsilon_1}, H^2_{\varepsilon_2}, \ldots, H^m_{\varepsilon_m}$ mulţimile corespunzătoare pentru $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_m$.

Cum $\{H^1_{\varepsilon_1}, H^2_{\varepsilon_2}, \ldots, H^m_{\varepsilon_m}\}$ este o familie finită de mulţimi, una dintre ele $H^{k_1}_{\varepsilon_{k_1}}$, va conţine o infinitate de elemente a şirului $\{x_n\}$. Notăm acest subşir cu $\{x_n^1\}$. Pentru ε_2 şi $\{x_n^1\}$ va exista o mulţime $H^{k_2}_{\varepsilon_{k_2}}$ care conţine un subşir al şirului $\{x_n^1\}$, notat cu $\{x_n^2\}$ şi $d(H^{k_2}_{\varepsilon_2}) < \varepsilon_2$.

Continuînd, vom găsi o mulţime $H^{k_n}_{\varepsilon_{k_n}}$ cu proprietatea că $d(H^{k_n}_{\varepsilon_{k_n}}) < \varepsilon_{k_n}$ şi care conţine o infinitate de termeni ai şirului $\{x_n\}$. Observăm că $H^{k_1}_{\varepsilon_{k_1}} \supset H^{k_2}_{\varepsilon_{k_2}} \supset \ldots \supset H^{k_2}_{\varepsilon_{k_n}} \supset \ldots$ şi că $\lim_n d(H^{k_n}_{\varepsilon_{k_n}}) = 0$. Cum l_2 este complet, rezultă că există $x_0 \in H^{k_n}_{\varepsilon_{k_n}}$, $n \in N$.

Dacă $\{x_n^m\} \in H^{k_m}_{\varepsilon_{k_m}}$ şi $\{x_n^m\} \subset \{x_n\}$ avem:

$$\left[\sum_{i=1}^{\infty}(x_n^{mi} - x_0^1)^2\right]^{1/2} \leqslant d(H^{k_m}_{\varepsilon_{k_m}}) < \varepsilon_{k_m}.$$

Deci şirul $\{x_n^m\}$ converge la $x_0 \in H$, şi cu aceasta am demonstrat compacitatea lui H.

P r o p o z i ţ i e. *Orice submulţime convexă şi compactă a lui H are proprietatea punctului fix.*

Demonstrăm mai întîi că H are proprietatea punctului fix. Fie f o aplicaţie continuă a lui H în el însuşi. Definim aplicaţiile

$$g_n : H \to H \text{ prin } g_n(x_k) = \begin{cases} x_k \text{ pentru } l \leqslant k \leqslant n \\ 0 \text{ pentru } k > n. \end{cases}$$

Se arată că $g_n(H) = H_n$ este homeomorf cu sfera unitate a lui R^n. Deci avem

$$\begin{tikzcd} H \arrow[rr, "f"] \arrow[dr, "g_n \circ f"'] & & H \arrow[dl, "g_n"] \\ & H_n & \end{tikzcd}$$

unde $g_n \circ f$ este o aplicaţie continuă a lui H în H_n. Observăm că

$$g_n \circ f|_{H_n} : H_n \to H_n.$$

Cum H_n este homeomorf cu o sferă din R^n deci şi cu un simplex n-dimensional, va exista, după teorema lui Brouwer, un şir $x_n \in H_n$ cu proprietatea că $(g_n \circ f)(x_n) = (x_n) \Leftrightarrow g_n(f(x_n)) = x_n$. Cum H este compact şi $x_n \in H_n \subset H$ putem presupune că $x_n \to x \in H_n$ Obţinem:

$$\| (g_n \circ f)(x_n) - f(x_n) \| = \left[\sum_{k=n+1}^{\infty} (-f(x_k))^2 \right]^{1/2} \leqslant \left(\sum_{k=n+1}^{\infty} \frac{1}{k^2} \right)^{1/2}.$$

Deoarece $\sum_{k=n+1}^{\infty} \frac{1}{k^2}$ este restul unei serii convergente, rezultă că $\| (g_n \circ f)(x_n) - f(x_n) \| \to 0$ şi prin urmare $\| g_n(f(x_n)) - f(x_n \| \to 0$. Deci luăm $\| g_n)f(x_n)) - f(x_n) \| = \| x - f(x) \| = 0$, de unde rezultă că $f(x) = x$ cu $x \in H_n$.

Demonstrăm acum că orice submulţime proprie a lui H are proprietatea punctului fix dacă este convexă şi compactă. Fie $C \subset H$, convexă şi compactă. Realizăm corespondenţa $x \to a(x)$ unde $a(x) \in C$ şi este elementul de cea mai bună aproximaţie a lui x. Deci $a : H \to C$ şi $\forall x \in H$ $\| x - a(x) \| \leqslant \| x - y \|$, $\forall y \in C$.

Arătăm că $a : H \to C$ este continuă. Presupunem că $\lim_n a(x_n) \neq a(x)$ cu toate că $x_n \to x$. Deoarece C este compactă, orice subşir $a(x_n^k)$ al lui $a(x_n)$ converge la $z \in C$, $z \neq a(x)$. Vom avea:

$$\| x_n^k - a(x_n^k) \| \leqslant \| x_n^k - a(x) \| \leqslant \| x_n^k - x \| + \| x - a(x) \|.$$

Am ţinut cont că $a(x_n^k)$ este elementul de cea mai bună aproximaţie pentru x_n^k. Trecînd la limită, obţinem:

$$\| x - z \| < \| x - x \| + \| x - a(x) \| = \| x - a(x) \|.$$

Aceasta contrazice proprietatea lui $a(x)$; deci: $a : H \to C$ este continuă.

Dacă $f : C \to C$ este continuă atunci $f \circ a : H \to C$, şi este de asemenea continuă. După cele arătate mai sus, există $x \in C$ cu $(f \circ a)(x) = x \Leftrightarrow f(a(x)) = x \in f(C) \subset C$. Dar pentru $x \in C$, $a(x) = x$ şi obţinem $f(x) = x$, adică şi C are proprietatea punctului fix.

Revenim la spaţiul (X, τ), unde topologia τ este generată de familia suficientă de seminorme $\{ s_i \mid i \in I \}$. Se poate arăta că orice mulţime compactă şi convexă $C \subset X$ are o submulţime proprie închisă şi convexă, C_1, cu proprietatea $f(C_1) = C_1$, unde $f : C \to C$ este o aplicaţie continuă. Se vede că C_1 este compactă, deoarece topologia este separată. Un caz particular ar fi acela cînd C şi C_1 sînt formate dintr-un număr finit de puncte.

Enunţăm teorema lui Tihonov. *Dacă (X, τ) este un spaţiu local convex şi topologia τ este generată de o familie suficientă de seminorme*

$\{ s_i \mid i \in I \}$, *atunci orice submulţime nevidă, compactă şi convexă a lui* X, *are proprietatea punctului fix.*

Demonstraţie. Din [2], amintim, lema 2: Dacă E este un spaţiu compact şi $\Gamma \subset E \times P(E)$ este o relaţie multivocă semicontinuă superior şi punctual compactă cu $\Gamma(x) \neq \emptyset$, $\forall x \in E$, atunci există un singur compact $K \subset E$ cu $\Gamma(K) = K$.

Din cele de mai sus rezultă că există $C_1 \subset C$, compactă şi convexă cu $f(C_1) \subset C_1$. Apoi, există $C_2 \subset C_1$, compactă şi convexă cu $f(C_2) \subset C_2$. Continuăm acest raţionament. Restricţia $f|_{\bigcap_{i=1}^{\infty} C_i} : \bigcap_{i=1}^{\infty} C_i \to \bigcap_{i=1}^{\infty} C_i$ este, după lema de mai sus o surjecţie. Cum $\bigcap_{i=1}^{\infty} C_i = \{ x_0 \}$, rezultă că $f(x_0) = x_0$ şi teorema este demonstrată.

Corolar 1. *Fie* X *un spaţiu local convex, separat Hausdorff, cu proprietatea că învelitoarea convexă închisă a oricărei mulţimi compacte, este compactă. Dacă* $C \subset X$, *este închisă şi convexă, iar* $f: C \to C$, *continuă, cu* $f(C)$ *relativ compactă în* X, *atunci* f *admite cel puţin un punct fix.*

Demonstraţie. Fie M învelitoarea convexă a mulţimii $\overline{f(A)}$ în X şi $\overline{M} = K \subset C$, nevidă, compactă şi convexă. Avem $f(K) \subset f(C) \subset K$, deoarece K este cea mai mică mulţime închisă şi convexă ce conţine pe $f(C)$. Conform ipotezelor, K este compactă, iar după teorema lui Tihonov, va exista un punct fix al lui f, $x_0 \in K \subset C$.

Corolar 2 (teorema I-a a lui Schauder). *Într-un spaţiu Banach, orice mulţime convexă şi compactă are proprietatea punctului fix.*

Demonstraţia este imediată.

Corolar 3 (teorema a II-a a lui Schauder). *Fie* X *un spaţiu Banach şi* $C \subset X$, *convexă şi închisă, iar* $f: C \to C$ *o aplicaţie continuă cu* $f(C)$ *compactă. Atunci,* f *are cel puţin un punct fix.*

Demonstraţie. Într-un spaţiu Banach, cea mai mică mulţime convexă ce conţine un compact, este compactă. Deci sîntem în condiţiile corolarului 1.

Corolar 4. *Într-un spaţiu normat de dimensiune finită, orice mulţime convexă şi compactă are proprietatea punctului fix.*

Demonstraţia este imediată, deoarece un astfel de spaţiu este complet.

Corolar 5 (teorema lui Brouwer). *În* R^n, *simplexele şi mulţimile homeomorfe cu ele au proprietatea punctului fix.*

Corolar 6. *Într-un spaţiu normat finit dimensional, mulţimea* $C = \{ x : \| x \| = 1 \}$ *nu este o retracţie a mulţimii* $K = \{ x : \| x \| \leqslant 1 \}$.

Demonstraţie. Spunem că C este o retracţie a lui K, dacă există $f: K \to C$, continuă şi cu proprietatea că $f(x) = x$, $\forall x \in C$. Vom demonstra imposibilitatea găsirii unei aplicaţii cu aceste două proprietăţi.

Într-adevăr, dacă ar exista $f: K \to C$ cu proprietăţile amintite, atunci aplicaţia $(-f): K \to C \subset K$ este o aplicaţie continuă a lui K în el însuşi care un are punct fix, ceea ce contrazice corolarul 5.

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UNE OBSERVATION SUR LES THÉORÈMES DE SCHAUDER

(Résumé)

On démontre le théorème de Tihonov, en employant des résultats de la téorie des multifonctions et on présent quelques corollaires, parmi lesquels les théorèmes de Schauder.

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A CHARACTERIZATION OF CERTAIN NONANALYTIC FUNCTIONS

BY

JOVAN D. KEČKIĆ

Complex functions which do not satisfy the Cauchy-Riemann equations have been studied for just over a century (the first trace of such functions appears in 1868 in a note by E. Beltrami [1]), different authors giving those functions different names : polygenic, generalized analytic, nonanalytic, etc. The next step was naturally to examine the second derivative of nonanalytic functions, and particularly to consider those functions which possess a unique second derivative (see, for example, [2], [3]). An attempt to give a short survey of the history of nonanalytic functions was made in [4].

The object of this note is to synthesize and characterize, from a different view-point, certain classes of nonanalytic functions, and in the light of these ideas to offer simple proofs and natural extensions of the results obtained by V. C. Poor [5]. Further on, $\boldsymbol{R}$ will, as usual denote the set of reals, and C the set of complex numbers.

We begin by noting that the concept of independence of two functions $f, g : \boldsymbol{R}^2 \to \boldsymbol{R}$ is easily extended to two functions $\varphi, \psi : \boldsymbol{R}^2 \to C$, and that the necessary and sufficient condition for the independence of φ, ψ is again $J \equiv \varphi_x \psi_y - \varphi_y \psi_x \neq 0$. In the special case $\varphi(x, y) = x + \mathrm{i}y$, $\psi(x, y) = x - \mathrm{i}y$, we have $J = 2\mathrm{i} \neq 0$, showing that z and $\bar{z}$ are independent functions.

We shall therefore write a complex function $f : C \to C$ in the form $f(z, \bar{z})$, and shall assume that it possesses partial derivatives with respect to z and $\bar{z}$ of the order needed.

It is readily verified that if $f(z, \bar{z}) = u(x, y) + \mathrm{i}v(x, y)$, then

$$\frac{\partial f}{\partial z} = \frac{1}{2}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \mathrm{i}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\right), \qquad \frac{\partial f}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} + \mathrm{i}\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)\right),$$

$$\frac{\partial^2 f}{\partial z^2} = \frac{1}{4}\left(\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} + 2\frac{\partial^2 v}{\partial x \partial y} + \mathrm{i}\left(\frac{\partial^2 v}{\partial x^2} - \frac{\partial^2 v}{\partial y^2} - 2\frac{\partial^2 u}{\partial x \partial y}\right)\right),$$

$$\frac{\partial^2 f}{\partial \bar{z}^2} = \frac{1}{4}\left(\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} - 2\frac{\partial^2 v}{\partial x \partial y} + \mathrm{i}\left(\frac{\partial^2 v}{\partial x^2} - \frac{\partial^2 v}{\partial y^2} + 2\frac{\partial^2 u}{\partial x \partial y}\right)\right),$$

$$\frac{\partial^2 f}{\partial z \partial \bar{z}} = \frac{1}{4}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \mathrm{i}\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right)\right), \qquad \text{etc.}$$

Then, two of the following statements are clearly equivalent:

$$\frac{\partial f}{\partial \bar{z}} = 0, \ f \text{ is an analytic function;}$$

$$(1) \qquad \frac{\partial^2 f}{\partial \bar{z}^2} = 0, \ f \text{ is an areolar monogenic function;}$$

$$\frac{\partial^2 f}{\partial z^2} = 0, \ f \text{ is a mean monogenic function.}$$

In virtue of (1), we see that the notation $f(z)$ should be reserved for analytic functions only, since they do not depend on $\bar{z}$. On the other hand, functions f satisfying $\frac{\partial f}{\partial z} = 0$ do not depend on z. They are analytic in $\bar{z}$, and shall be called c - analytic functions (from conjugate analytic functions). In the sequel $a_1(z)$, $a_2(z)$,... will denote arbitrary analytic functions, and $c_1(\bar{z})$, $c_2(\bar{z})$, ... arbitrary c - analytic functions.

Theorems 2 and 4 from [5] easily follow from the above considerations. The following set of equivalences

$$f \text{ is areolar monogenic} \Leftrightarrow \frac{\partial^2 f}{\partial \bar{z}^2} = 0 \Leftrightarrow f = \bar{z} a_1(z) + a_2(z)$$

$$f \text{ is mean monogenic} \Leftrightarrow \frac{\partial^2 f}{\partial z^2} = 0 \Leftrightarrow f = z c_1(\bar{z}) + c_2(\bar{z})$$

provide natural proofs of Theorems 3 and 5 from [5]. Similarly, Theorem 6 from [5] follows on the observation that the general solution of the system of partial differential equations

$$\frac{\partial^2 f}{\partial z^2} = 0, \qquad \frac{\partial^2 f}{\partial \bar{z}^2} = 0$$

is given by $f(z, \bar{z}) = az\bar{z} + bz + c\bar{z} + d$, where a, b, c, d are arbitrary constants.

If we call the functions satisfying $\frac{\partial^{n+1} f}{\partial \bar{z}^{n+1}} = 0$ areolar monogenic functions of order n, and those satisfying $\frac{\partial^{n+1} g}{\partial z^{n+1}} = 0$ mean monogenic of order n, we see that:

(i) they are of the form

$$f(z, \bar{z}) = \sum_{\nu=0}^{n} \bar{z}^{\nu} a_{\nu}(z), \qquad g(z, \bar{z}) = \sum_{\nu=0}^{n} z^{\nu} c_{\nu}(\bar{z});$$

(ii) if f is areolar monogenic of order n, and at the same time mean monogenic of order n, then

$$f(z, \bar{z}) = \sum_{\nu=0}^{n} P_{\nu}(z) \bar{z}^{\nu} = \sum_{\nu=0}^{n} Q_{\nu}(\bar{z}) z^{\nu},$$

where P_{ν}, $Q_{\nu} (\nu = 0, 1, \ldots, n)$ are arbitrary polynomials of degree n.

(iii) there is a complete duality between those two classes of functions, in the sense that every function of one class has an exact correspondent in the other.

This can be further generalized. Let a class of complex functions $f(z, \bar{z})$ be defined as the general solution of the partial differential equation

$$E\left((z, \bar{z}, f, \frac{\partial f}{\partial z}, \frac{\partial f}{\partial \bar{z}}, \frac{\partial^2 f}{\partial z^2}, \frac{\partial^2 f}{\partial \bar{z}^2}, \frac{\partial^2 f}{\partial z \partial \bar{z}}, \ldots, \frac{\partial^n f}{\partial \bar{z}^n}\right) = 0.$$

Then there is a complete duality between this class of functions and the class of complex functions g defined as the general solution of the partial differential equation

$$E\left(\bar{z}, z, g, \frac{\partial g}{\partial \bar{z}}, \frac{\partial g}{\partial z}, \frac{\partial^2 g}{\partial \bar{z}^2}, \frac{\partial^2 g}{\partial z^2}, \frac{\partial^2 g}{\partial \bar{z} \partial z}, \ldots, \frac{\partial^n g}{\partial z^n}\right) = 0.$$

So, for example, the class of complex functions generated by the equation

$$\frac{\partial^{2n} f}{\partial z^n \partial \bar{z}^n} = 0$$

is self dual. Those functions are of the form

$$f(z, \bar{z}) = \sum_{\nu=0}^{n-1} (\bar{z}^{\nu} a_{\nu}(z) + z^{\nu} c_{\nu}(\bar{z})).$$

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O CARACTERIZARE A ANUMITOR FUNCŢII NEANALITICE

(Rezumat)

Plecînd de la faptul că z şi $\bar{z}$ sînt funcţii independente de x, y, se descriu anumite funcţii neanalitice ca soluţii de ecuaţii cu derivate parţiale în raport cu z şi $\bar{z}$. Se dau de asemenea demonstraţii simple ale unor rezultate obţinute de V. C. Poor.

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ORTHOGONAL POLYNOMIALS IN A CERTAIN CLASS OF POLYNOMIALS [1])

BY

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1. The problem of characterizing orthogonal polynomial sets which satisfy certain generating relations has been of interest to several mathematicians. For example orthogonal polynomial sets which are also Appell have been studied by Angelesco [4], Carlitz [5], Shohat [8], Toscano [9] and Webster [10].

Meixner [6] and later Sheffer [7] studied orthogonal polynomial sets $\{p_n(x)\}_0^\infty$ having the generating relation

$$A(t)\, e^{xH(t)} = \sum_0^\infty p_n(x) t^n, \tag{1}$$

where $A(t)$ and $H(t)$ are (at least formal) power series and

$$A(t) = \sum_0^\infty a_n t^n \qquad (a_0 \neq 0), \tag{2}$$

$$H(t) = \sum_0^\infty h_n t^n \qquad (h_1 \neq 0). \tag{3}$$

They calculated all orthogonal polynomial sets in this class of polynomials. We shall refer to these polynomial sets as Meixner's polynomials.

Al-Salam [2] studied polynomial sets $\{p_n(x)\}_0^\infty$ having the generating relation

$$e^{\Phi(x,t)} = \sum_0^\infty p_n(x) t^n,$$

[1]) This paper is part of the thesis written for the M. Sc. by the author under the supervision of Professor W. A. Al-Salam.

where

$$\Phi(x, t) = \sum_{j=0}^{h} x^j A_j(t), \quad (k \geqslant 1),$$

and

$$A_j(t) = \sum_{r=0}^{\infty} a_{jr} t^r, \quad (j = 0, 1, \ldots, k),$$

and proved that the Meixner's polynomials are the only orthogonal polynomials in this class of polynomials.

2. In this note we study orthogonal polynomial sets $\{p_n(x)\}_0^\infty$ which are generated by

$$A(t) e_q(xH(t)) = \sum_0^\infty p_n(x) t^n, \tag{4}$$

where

$$e_q(w) = 1 + \sum_1^\infty \frac{w^n (1-q)^n}{(q)_n}, \quad (q \neq 1), \quad e_1(w) = e^w,$$

$$(q_n) = (1-q)(1-q^2)\ldots(1-q^n), \qquad (n \geqslant 1),$$

and $H(t)$ and $A(t)$ satisfy (2) and (3) respectively. There is no loss of generality to assume that $a_0 = h_1 = 1$.

Theorem. *The only orthogonal polynomial sets generated by* (4) *such that* $H(t)$ *and* $A(t)$ *satisfy* (2) *and* (3) *respectively are:*

(i) *Meixner's polynomials if* $q = 1$.

(ii) *The polynomials* $\{U_n^a(x)\}_0^\infty$ *and* $\{V_n^a(x)\}_0^\infty$ *of Al-Salam and Carlitz* [3] *if* $q(1-q) \neq 0$ *and* $q^n \neq 1$ *for* $n = 1, 2, 3, \ldots$

(iii) *The set of Tchebicheff polynomials of the second kind* $\{U_n(x)\}_0^\infty$ *and the polynomials* $p_n(x; s, a)$ *studied by W. Allaway,* [1], *if* $q = 0$.

Before we prove this theorem we shall first establish the following lemma.

Lemma. *If* $q(1-q) \neq 0$ *and* $\{p_n(x)\}_0^\infty$ *is a set of orthogonal polynomials satisfying* (2), (3) *and* (4), *then* $H(t) = t$.

Proof. We introduce the q-difference operator

$$D_q f(x) = \frac{f(qx) - f(x)}{(q-1)x}, \quad (q \neq 1).$$

We note that

$$D_q f(x)\, g(x) = f(qx)\, D_q g(x) + g(x)\, D_q f(x). \tag{5}$$

Since $D_q e_q(xH(t)) = H(t)\, e_q(xH(t))$, it follows that (4) holds if and only if

$$D_q\, p_n(x) = \sum_{h=1}^{n} h_k\, p_{n-k}(x), \tag{6}$$

where the h_k's are the same as in (3).

We next recall that a polynomial set $\{p_n(x)\}_0^\infty$ is orthogonal if and only if there exist constants B_n and C_n such that

$$\frac{1}{\alpha_{n+1}} p_{n+1}(x) = \frac{1}{\alpha_n}(x - B_n)\, p_n(x) - \frac{C_n}{\alpha_{n+1}} p_{n-1}(x), \qquad (n \geqslant 0) \tag{7}$$

where $p_{-1}(x) = 0$, $p_0(x) = 1$ and α_n is the leading coefficient in $p_n(x)$. Thus $\alpha_0 = 1$ and

$$\alpha_n = \frac{(1-q)^n}{(q)_n}. \tag{8}$$

Substituting for α_n from (8) into (7), applying D_q to both sides of the resulting equation, making use of (5) and (6) we get

$$\begin{aligned} \frac{1-q^{n+1}}{1-q} \sum_{k=1}^{n+1} h_k p_{n+1-k}(x) &= q \sum_{k=1}^{n} h_k \Big[B_{n-k}\, p_{n-k}(x) + \\ &+ \frac{1-q^{n-k+1}}{1-q} p_{n-k+1}(x) + C_{n-k} \frac{1-q}{1-q^{n-k}} p_{n-k-1}(x) \Big] + \\ &+ p_n(x) - B_n \sum_{k=1}^{n} h_k p_{n-k}(x) - \frac{1-q}{1-q^n} C_n \sum_{k=1}^{n-1} h_k p_{n-k-1}(x). \end{aligned} \tag{9}$$

Equating coefficients of $p_{n-1}(x)$ in both sides of (9) we get a linear difference equation in B_n whose solution is

$$B_n = Bq^n - h_2 \frac{1-q^n}{1-q} - nh_2 q^n, \qquad (n = 1, 2, \ldots), \tag{10}$$

where B is a periodic function if n of period 1.

Equating coefficients of $p_{n-2}(x)$ in both sides of (9), substituting for B_n from (10) we get a linear difference equation in C_n whose solution is

$$\begin{aligned} \frac{1-q}{1-q^n} C_n &= C_n + (h_2^2 - h_3) \frac{1-q^n}{1-q} + \\ &+ n\{h_2 B(1-q) + 3h_2^2 - (1+q)\, h_3\}\, q^{n-1}, \qquad (n > 1). \end{aligned} \tag{11}$$

Equating coefficients of $p_{n-k-1}(x)$ in (9) we get

$$\begin{aligned} &h_{k+2}\left[\frac{1-q^{n+1}}{1-q} - q\, \frac{1-q^{n-k-1}}{1-q} \right] + h_{k+1}\,[B_n - qB_{n-k-1}] + \\ &+ h_k \left\{ \frac{1-q}{1-q_n} C_n - q\, \frac{1-q}{1-q^{n-k}} C_{n-k} \right\} = 0. \end{aligned} \tag{12}$$

Writing $n+1$ for n in (12), subtracting q times (12) from the resulting equation and using (10) and (11) we get

$$(1-q)\,h_{k+2}-h_2h_{k+1}(1+q^{n+1}-q-q^{n-k})+h_k\,[(h_2^2-h_3)\,(1-q)+$$

$$(13)\qquad +\{h_2\,B(1-q)+3h_2^2-(1+q)h_3\}\,(q^n-q^{n-k+1})-$$

$$-(n+1)\,(1-q)h_2^2\,(q^n-q^{n-k+1})-k(1-q)h_2^2q^{n-k+1}]=0\,.$$

Writing $n+1$ for n in (13) and subtracting (13) form the resulting equation and dividing by $(1-q)\,q^n$ we get

$$h_2h_{k+1}\left\{q-\frac{1}{q^k}\right\}+h_k\left[\{h_2\,B(1-q)+3h_2^2-(1+q)\,h_3\}\left\{\frac{q}{q^k}-1\right\}+\right.$$

$$\left.+k(1-q)h_2^2\frac{q}{q^k}-qh_2^2\left\{1-\frac{q}{q^k}\right\}+(n+1)\,h_2^2\left\{1-\frac{q}{q^k}\right\}\right]=0\,.$$

As this is an identity in n we get

$$(14)\qquad h_k=0,\qquad (k=2,\ 3\ ,...),$$

and the result follows.

Corollary. *Under conditions of the lemma, the B_n's and C_n's are given by*

$$(15)\qquad \begin{cases} B_n=Bq^n, & (n=1\ ,...),\\ \dfrac{1-q}{1-q^n}\,C_n=Cq^n, & (n=2\ ,...).\end{cases}$$

In fact B_0 and C_1 are arbitrary.

Proof of (ii). Substituting for h_k, B_n and C_n from (14) and (15) into (7) we get

$$(16.a)\qquad \frac{1-q^{n+1}}{1-q}\,p_{n+1}(x)=(x-Bq^n)\,p_n(x)-Cq^np_{n+1}(x),\qquad (n=2,3\ ,...),$$

$$(16.b)\qquad \frac{1-q^2}{1-q}\,p_2(x)=(x-Bq)p_1(x)-(Cq+E)p_0(x),$$

$$(16.c)\qquad \frac{1-q}{1-q}\,p_1(x)=(x-B-F)p_0(x),$$

where $B_0=B+F$, $C_1=Cq+E$, E and F are arbitrary constants. Multiplying (16.a), (16.b), (16.c) by t^{n+1}, t^2, t respectively and adding for $n=2,\ 3\ ,...$ we get

$$\frac{A(t)\,e_q(xt)-A(qt)\,e_q(xqt)}{1-q}=xtA(t)\,e(xt)-BtA(qt)e_q(xqt)-$$

$$-Ct^2\,qA(qt)\,e(xqt)-Ft-Et^2.$$

Making use of

$$e_q(qw) = \{1 - (1 - q)\, w\} e_q(w),$$

we get

$$e_q(qxt)\,[A(t) - A(qt)\{1 - Bt(1 - q) - Ct^2 q(1 - q)\}] = -(Ft + Et^2)(1 - q).$$

Therefore $F = E = 0$, and

$$A(t) = A(qt)\,(1 - \alpha t)\,(1 - \beta t), \tag{17}$$

where

$$1 - B(1 - q)t - Cq(1 - q)t^2 = (1 - \alpha t)(1 - \beta t).$$

whe solution of (17) is given by

$$A(t) = \left\{ e_q\left(\frac{\alpha t}{1 - q}\right) e_q\left(\frac{\beta t}{1 - q}\right) \right\}^{-1},$$

and the result follows.

P r o o f of (iii). In this case we have

$$B_n = B, \quad (n \geqslant 1); \quad B_0 = E + B, \tag{18}$$

$$C_n = C, \quad (n \geqslant 2); \quad C_1 = C - F, \tag{19}$$

There B, C, E and F, are constants

Therefore, we have

$$p_1(x) = x - B - E,$$

$$p_2(x) = (x - B)p_1(x) + (F - C)p_0(x),$$

$$p_{n+1}(x) = (x - B)p_n(x) - Cp_{n-1}(x), \quad (n < 1).$$

Let

$$G(x, t) = \Sigma p_n(x)\, t^n,$$

therefore (20) gives

$$G(x, t) = \frac{1 - Et + Ft^2}{1 - (x - B)\,t + Ct^2}. \tag{21}$$

W. A l l a w a y [1] studied polynomial sets $\{p_n(x\,;\, s,\, \alpha)\}_0^\infty$ defined by

$$\sum_0^\infty p_n(x\,;\, s,\, \alpha)\, t^n = \frac{1 - st + t^2(1 - \alpha)}{1 - xt + t^2}, \tag{22}$$

and hence

$$p_n(x) = p_n(x;\ E,\ 1-F).$$

The case $E = F = 0$ gives the Tchebycheff polynomials.

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POLINOAME ORTOGONALE ÎNTR-O ANUMITĂ CLASĂ DE POLINOAME

(Rezumat)

Se caracterizează toate polinoamele ortogonale $\{p_n(x)\}_0^\infty$ generate de (4), în care $e_q(x)$ te q-analoga funcţiei exponenţiale iar $A(t)$ şi $H(t)$ sînt serii de puteri în t.

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ON SOME PROPERTIES OF GENERALISED HANKEL TRANSFORM (II)

BY

R. U. VERMA

1. Introduction

Recently the author (1969) has given a generalisation of Hankel transform expressed by the integral equation

$$f(x, y) = \int_0^\infty \int_0^\infty \overset{2,1}{\underset{2,4}{G}} \left(xu \left| \begin{matrix} k - m - \frac{1}{2} - \frac{\nu}{2}, & -k + m + \frac{1}{2} + \frac{\nu}{2} \\ \frac{\nu}{2} - \lambda - m, \frac{\nu}{2} - \lambda + m, & -\frac{\nu}{2} + \lambda + m, \ -\frac{\nu}{2} + \lambda - m \end{matrix} \right. \right) \times$$

$$\times \overset{2,1}{\underset{2,4}{G}} \left(yu' \left| \begin{matrix} k' - m' - \frac{1}{2} - \frac{\nu'}{2}, & -k' + m' + \frac{1}{2} + \frac{\nu'}{2} \\ \frac{\nu'}{2} - \lambda' - m', \frac{\nu'}{2} - \lambda' + m', & -\frac{\nu'}{2} + \lambda' + m', \ -\frac{\nu'}{2} + \lambda' - m' \end{matrix} \right. \right) \times \tag{1.1}$$

$$\times\ g(u, u')\, du\, du',$$

where m, m' are neither integers nor zero.

This generalisation is an extension of the generalisation introduced by Verma (1968). We call $f(x, y)$ as $\Phi_{\nu;\nu'}(k, \lambda, m; k, \lambda', m')$ - transform.

In this note some theorems connected with the above generalisation are investigated and latter these are illustrated by the example.

2. Theorem 1. *If* $1 \leqslant p \leqslant 2$, $j = 1, 2$, $f_j(t, t_1)$ *belongs to* $L_p(0, \infty)$ *and*

$$F_j(x, y) = \int_0^\infty \int_0^\infty \overset{2,1}{\underset{2,4}{G}} \left(xu \left| \begin{matrix} k - m - \frac{1}{2} - \frac{\nu}{2}, & -k + m + \frac{1}{2} + \frac{\nu}{2} \\ \frac{\nu}{2} - \lambda - m, \frac{\nu}{2} - \lambda + m, & -\frac{\nu}{2} + \lambda + m, \; -\frac{\nu}{2} + \lambda - m \end{matrix} \right. \right) \times$$

$$(2.1) \quad \times \overset{2,1}{\underset{2,4}{G}} \left(yu' \left| \begin{matrix} k' - m' - \frac{1}{2} - \frac{\nu'}{2}, & -k' + m' + \frac{1}{2} + \frac{\nu'}{2} \\ \frac{\nu'}{2} - \lambda' - m', \frac{\nu'}{2} - \lambda' + m', & -\frac{\nu'}{2} + \lambda' + m', \; -\frac{\nu'}{2} + \lambda' - m' \end{matrix} \right. \right) \times$$

$$\times f_j(u, u') \, du \, du',$$

then

$$(2.2) \quad (rr_1)^{-1} \int_0^\infty \int_0^\infty F_1\left(\frac{x}{r}, \frac{y}{r_1}\right) f_2(sx, s, y) dx dy = (ss_1)^{-1} \int_0^\infty \int_0^\infty F_2\left(\frac{x}{s}, \frac{y}{s_1}\right) f_1(rx, r, y) \, dx \, dy,$$

provided the integrals exist.

P r o o f. Taking left side of (2.2), we have

$$(rr_1)^{-1} \int_0^\infty \int_0^\infty F_1\left(\frac{x}{r}, \frac{y}{r_1}\right) f_2(sx, s, y) \, dx \, dy =$$

$$= \int_0^\infty \int_0^\infty F_1(u, u') f_2(rsu, r, s, u') \, du \, du' =$$

$$= \int_0^\infty \int_0^\infty f_2(rsu, r, s, u') \int_0^\infty \int_0^\infty \overset{2,1}{\underset{2,4}{G}} \left(uv \left| \begin{matrix} k - m - \frac{1}{2} - \frac{\nu}{2}, & -k + m + \frac{1}{2} + \frac{\nu}{2} \\ \frac{\nu}{2} - \lambda - m, \frac{\nu}{2} - \lambda + m, & -\frac{\nu}{2} + \lambda + m, \; -\frac{\nu}{2} + \lambda - m \end{matrix} \right. \right) \times$$

$$\times \overset{1,2}{\underset{2,4}{G}} \left(u'v' \left| \begin{matrix} k' - m' - \frac{1}{2} - \frac{\nu'}{2}, & -k' + m' + \frac{1}{2} + \frac{\nu'}{2} \\ \frac{\nu'}{2} - \lambda' - m', \frac{\nu'}{2} - \lambda' + m', & -\frac{\nu'}{2} + \lambda' + m', \; -\frac{\nu'}{2} + \lambda' - m' \end{matrix} \right. \right) f_1(v, v') \, dv dv' du du' =$$

$$= \int_0^\infty \int_0^\infty f_1(v, v') dv dv' \int_0^\infty \int_0^\infty \overset{2,1}{\underset{2,4}{G}} \left(uv \left| \begin{matrix} k - m - \frac{1}{2} - \frac{\nu}{2}, & -k' + m' + \frac{1}{2} + \frac{\nu'}{2} \\ \frac{\nu}{2} - \lambda - m, & \frac{\nu}{2} - \lambda + m, \; -\frac{\nu}{2} + \lambda + m, -\frac{\nu}{2} + \lambda - m \end{matrix} \right. \right) \times$$

$$\times \underset{2,4}{\overset{2,1}{G}}\left(u'v' \left| \begin{matrix} k'-m'-\frac{1}{2}-\frac{\nu'}{2}, & -k'+m'+\frac{1}{2}+\frac{\nu'}{2} \\ \frac{\nu'}{2}-\lambda'-m', \frac{\nu'}{2}-\lambda'+m', & -\frac{\nu'}{2}+\lambda'+m', -\frac{\nu'}{2}+\lambda'-m' \end{matrix} \right. \right) \times$$

$$\times f_2(rsu,\ r,\ s,\ u')\,\mathrm{d}u\,\mathrm{d}u' =$$

$$= (rr_1\ ss_1)^{-1}\int_0^\infty\int_0^\infty f_1(v,\ v')\,F_2\left(\frac{v}{rs},\ \frac{v'}{r_1s_1}\right)\mathrm{d}v\,\mathrm{d}v' =$$

$$= (ss_1)^{-1}\int_0^\infty\int_0^\infty f_1(rx,\ r,\ y)\,F_2\left(\frac{x}{s},\ \frac{y}{s_1}\right)\mathrm{d}x\,\mathrm{d}y,$$

which proves the theorem.

Assuming the integrals in (2.1) are absolutely convergent, the change in the order of integration will be justified, if the integrals in (2.2) are absolutely convergent.

In particular, if we put $r = 1$, $s = 1$, $r_1 = 1$, $s_1 = 1$

$$\int_0^\infty\int_0^\infty F_1(x,\ y)\,f_2(x,\ y)\,\mathrm{d}x\,\mathrm{d}y = \int_0^\infty\int_0^\infty F_2(x,\ y)\,f_1(x,\ y)\,\mathrm{d}x\,\mathrm{d}y, \tag{2.3}$$

which may be called the Parseval's theorem for the generalised Hanke transform $\Phi_{\nu;\,\nu'}(k,\ \lambda,\ m\,;\ k',\ \lambda',\ m')$ - transform [3] of two variables.

Theorem 2. *If $f(x,\ y)$ and $g(x,\ y)$ are $\Phi_{\nu;\,\nu'}(k,\ \lambda,\ m\,;\ k'\ \lambda',\ m')$ - transforms and $f_1(r,\ s)$ and $g_1(r,\ s)$ are the Mellin transforms of $f(x,\ y)$ and $g(x,\ y)$, then*

$$\frac{f_1(r,\ s)}{g_1(1-r, 1-s)} = \frac{\Gamma\left(\frac{\nu}{2}-\lambda+m+r\right)\Gamma\left(\frac{\nu}{2}-\lambda-m+r\right)\Gamma\left(\frac{1}{2}-k+m-\frac{\nu}{2}-r\right)}{\Gamma\left(\frac{\nu}{2}-k+m-\frac{1}{2}+r\right)\Gamma\left(\frac{\nu}{2}+1-\lambda+m-r\right)\Gamma\left(\frac{\nu}{2}+1-\lambda-m-r\right)}\times$$

$$\times\frac{\Gamma\left(\frac{\nu'}{2}-\lambda'+m'+s\right)\Gamma\left(\frac{\nu'}{2}-\lambda'-m'+s\right)\Gamma\left(\frac{1}{2}-k'+m'+\frac{\nu'}{2}-s\right)}{\Gamma\left(\frac{\nu'}{2}-k'+m'-\frac{1}{2}+s\right)\Gamma\left(\frac{\nu'}{2}+1-\lambda'+m'-s\right)\Gamma\left(\frac{\nu'}{2}+1-\lambda'-m'-s\right)},$$

provided

$$R(\nu + 2m - 2\lambda) > 0, \quad R\left(\frac{1}{2} - k + m + \frac{\nu}{2}\right) > 0, \quad R(\nu - 2\lambda - 2m) > 0,$$

$$R(\nu' + 2m' - 2\lambda) > 0, \quad R\left(\frac{1}{2} - k' + m' + \frac{\nu'}{2}\right) > 0,$$

$$R(\nu' - 2\lambda' - 2m') > 0.$$

Proof. $$f(x,y) = \int_0^\infty\int_0^\infty G^{2,1}_{2,4}\left(xu \left| \begin{matrix} k - m - \frac{1}{2} - \frac{\nu}{2}, & -k + m + \frac{1}{2} + \frac{\nu}{2} \\ \frac{\nu}{2} - \lambda - m, \frac{\nu}{2} - \lambda + m, & -\frac{\nu}{2} + \lambda + m, -\frac{\nu'}{2} + \lambda - m \end{matrix}\right.\right) \times$$

$$\times G^{2,1}_{2,4}\left(yu' \left| \begin{matrix} k' - m' - \frac{1}{2} - \frac{\nu'}{2}, & -k' + m' + \frac{1}{2} + \frac{\nu'}{2} \\ \frac{\nu'}{2} - \lambda' - m', & \frac{\nu'}{2} - \lambda' + m', -\frac{\nu'}{2} + \lambda' + m', -\frac{\nu'}{2} + \lambda' - m' \end{matrix}\right.\right) g(u, u')\, du\, du' \tag{2.4}$$

and

$$g(u, u') = \frac{1}{(2\lambda i)^2} \int_{c - i\infty}^{c + i\infty} \int_{c' - i\infty}^{c' + i\infty} g_1(r, s) u^{-r} u'^{-s}\, dr\, ds, \tag{2.5}$$

where $r = \sigma + it$, $s = \gamma + i\tau$,

$$g_1(r, s) = O(e^{-k[|t| + |\tau|]}), \qquad k > 0, \text{ and } |t|, |r| \to \infty, \tag{2.6}$$

and σ and γ are the values of c and c' in the strips $0 < c < 1$ and $0 < c' < 1$.

Therefore,

$$f(x, y) = \int_0^\infty\int_0^\infty G^{2,1}_{2,4}\left(xu \left| \begin{matrix} k - m - \frac{1}{2} - \frac{\nu}{2}, & -k + m + \frac{1}{2} + \frac{\nu}{2} \\ \frac{\nu}{2} - \lambda - m, & \frac{\nu}{2} - \lambda + m, -\frac{\nu}{2} + \lambda + m, -\frac{\nu}{2} + \lambda - m \end{matrix}\right.\right) \times$$

$$\times G^{2,1}_{2,4}\left(yu' \left| \begin{matrix} k' - m' - \frac{1}{2} - \frac{\nu'}{2}, & -k' + m' + \frac{1}{2} + \frac{\nu'}{2} \\ \frac{\nu'}{2} - \lambda' - m', & \frac{\nu'}{2} - \lambda' + m', -\frac{\nu'}{2} + \lambda' + m', -\frac{\nu'}{2} + \lambda' - m' \end{matrix}\right.\right) du\, du' \times$$

$$\times \frac{1}{(2\lambda i)^2} \int_{c - i\infty}^{c + i\infty} \int_{c' - i\infty}^{c' + i\infty} g_1(r, s)\, u^{-r} u^{-s}\, dr\, ds = \tag{2.7}$$

$$= \frac{1}{(2\lambda i)^2} \int_{c-i\infty}^{c+i\infty} \int_{c'-i\infty}^{c'+i\infty} \frac{\Gamma\left(\frac{\nu}{2} - \lambda + m + r\right)\Gamma\left(\frac{\nu}{2} - \lambda - m + r\right)\Gamma\left(\frac{1}{2} - k + m + \frac{\nu}{2} - r\right)}{\Gamma\left(\frac{\nu}{2} - k + m + \frac{1}{2} + r\right)\Gamma\left(\frac{\nu}{2} + 1 - \lambda + m - r\right)\Gamma\left(\frac{\nu}{2} + 1 - \lambda - m - r\right)} \times$$

$$\times \frac{\Gamma\left(\frac{\nu'}{2} - \lambda' + m' + s\right)\Gamma\left(\frac{\nu'}{2} - \lambda' - m' + s\right)\Gamma\left(\frac{1}{2} - k' + m' + \frac{\nu'}{2} - s\right)}{\Gamma\left(\frac{\nu'}{2} - k' + m' + \frac{1}{2} + s\right)\Gamma\left(\frac{\nu'}{2} + 1 - \lambda' + m' - s\right)\Gamma\left(\frac{\nu'}{2} + 1 - \lambda' - m' - s\right)} \times$$

$$\times g_1(1 - r, 1 - s)\, x^{-r} y^{-s}\, dr\, ds,$$

provided the change in the order of integration is permissible. Hence, we obtain

$$f_1(r,s) = \frac{\Gamma\left(\frac{\nu}{2} - \lambda + m + r\right)\Gamma\left(\frac{\nu}{2} - \lambda - m + r\right)\Gamma\left(\frac{1}{2} - k + m + \frac{\nu}{2} - r\right)}{\Gamma\left(\frac{\nu}{2} - k + m + \frac{1}{2} + r\right)\Gamma\left(\frac{\nu}{2} + 1 - \lambda + m - r\right)\Gamma\left(\frac{\nu}{2} + 1 - \lambda - m - r\right)} \times$$

$$\times \frac{\Gamma\left(\frac{\nu'}{2} - \lambda' + m' + s\right)\Gamma\left(\frac{\nu'}{2} - \lambda' - m' + s\right)\Gamma\left(\frac{\nu'}{2} - k' + m' + s + \frac{\nu}{2}\right)}{\Gamma\left(\frac{\nu'}{2} - k' + m' + \frac{1}{2} + s\right)\Gamma\left(\frac{\nu'}{2} + 1 - \lambda' + m' - s\right)\Gamma\left(\frac{\nu'}{2} + 1 - \lambda' - m' - s\right)} \times$$

$$\times g_1(1 - r, 1 - s)\ ,$$

which proves the theorem.

The change in the order of integration is justified since the integrals converge absolutely and the repeated integrals exist.

Corollary 1. *By putting* $\lambda = -m$, $\lambda' = -m'$ *and latter* $k + m = \frac{1}{2}$, $k' + m' = \frac{1}{2}$ *and if* $f_1(r, s)$ *and* $g_1(r, s)$ *are Mellin transforms of* $f(x, y)$ *and* $g(x, y)$ *then we get*

$$\frac{f_1(r, s)}{g_1(1 - r, 1 - s)} = \frac{\Gamma\left(\frac{\nu}{2} + r\right)\Gamma\left(\frac{\nu'}{2} + s\right)}{\Gamma\left(\frac{\nu}{2} + 1 - r\right)\Gamma\left(\frac{\nu'}{2} + 1 - s\right)}.$$

Example. Let us consider the following pair of $\Phi_{\nu;\nu'}(k, \lambda, m; k', \lambda', m')$ - transform when $f(x, y) = g(x, y)$ and

$$f(x, y) = \overset{2,0}{\underset{1,2}{G}}\left(x \left| \begin{matrix} \frac{\nu}{2} - k + m + \frac{1}{2} \\ \frac{\nu}{2} - \lambda + m, \quad \frac{\nu}{2} - \lambda - m \end{matrix} \right.\right) \overset{2,0}{\underset{1,2}{G}}\left(y \left| \begin{matrix} \frac{\nu'}{2} - k' + m' + \frac{1}{2} \\ \frac{\nu'}{2} - \lambda' + m', \frac{\nu'}{2} - \lambda' - m' \end{matrix} \right.\right),$$

then

$$\frac{f_1(r, s)}{g_1(1-r, 1-s)} = \frac{\Gamma\left(\frac{\nu}{2} - \lambda + m + r\right)\Gamma\left(\frac{\nu}{2} - \lambda - m + r\right)}{\Gamma\left(\frac{\nu}{2} - k + m + \frac{1}{2} + r\right)\Gamma\left(\frac{1}{2}\nu + 1 - \lambda + m - r\right)} \times$$

$$\times \frac{\Gamma\left(\frac{1}{2} - k + m + \frac{\nu}{2} - r\right)\Gamma\left(\frac{1}{2}\nu' - \lambda' + m' + s\right)}{\Gamma\left(\frac{1}{2}\nu + 1 - \lambda - m - r\right)\Gamma\left(\frac{1}{2}\nu' - k' + m' + \frac{1}{2} + s\right)} \times$$

$$\times \frac{\Gamma\left(\frac{1}{2}\nu' - \lambda' - m' + s\right)\Gamma\left(\frac{1}{2} - k' + m' + \frac{1}{2}\nu' - s\right)}{\Gamma\left(\frac{1}{2}\nu' + 1 - \lambda' + m' - s\right)\Gamma\left(\frac{1}{2}\nu' + 1 - \lambda' - m' - s\right)},$$

provided

$$R(\nu + 2m - 2\lambda) > 0,$$

$$R\left(\frac{3}{2} - k + m + \frac{1}{2}\nu\right) > 0,$$

$$R(\nu - 2\lambda - 2m) > 0,$$

$$R(\nu' + 2m' - 2\lambda) > 0,$$

$$R\left(\frac{3}{2} - k' + m' + \frac{1}{2}\nu'\right) > 0$$

and

$$R(\nu' - 2\lambda' - 2m') > 0.$$

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ASUPRA UNOR PROPRIETĂŢI ALE TRANSFORMATELOR HANKEL GENERALIZATE (II)

(Rezumat)

Recent autorul a generalizat transformata Hankel exprimată prin ecuaţia integrală (1.1). În lucrarea de faţă se prezintă unele teoreme legate de această generalizare, care se ilustrează pe un exemplu.

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ON SOME FUNCTIONS OF A HYPERCOMPLEX VARIABLE

BY

ALFRED BRAIER

1. Let recall some previous results, established in [1]. Suppose that

$$p(\lambda) = \lambda^n + \alpha_1\lambda^{n-1} + \dots + \alpha_{n-1}\lambda \tag{1}$$

($\alpha \in R$) is the minimal polynomial of an element v of an associative and commutative algebra A_n on the real field, with unity element v^o. Let choose as basis in A_n the succesive powers of v, namely v^n, where $n = 0, 1, \dots, n-1$, with $v_1 = v^o$, i. e.

$$v_k = v^{k-1}, \quad (k = 1,2,\dots,n), \quad \text{and} \quad v^n = -\sum_{i=2}^{n} \alpha_i v^{n-i}. \tag{2}$$

Then, the components in the cyclical basis generated by v, of function $\exp(xv)$, ($x \in R$), are fundamental solutions of the differential equation

$$\varphi^{(n)} + \alpha_1\varphi^{(n-1)} + \dots + \alpha_{n-1}\varphi' + \alpha_n\varphi = 0 \tag{3}$$

having (1) as characteristic polynominal. For detalis and apllications see [1].

In this paper we shall prove that functions of a hypercomplex variable defined using element v which generates the cyclical basis, are of special interest in the field of partial differential equations.

2. Let remark that from (2_1) it follows

$$v^k/v = v^{k-1} \quad (k = 1, 2, \dots, n).$$

We shall need immediately the quotient $1/v$. It can be expressed in the cyclical basis

$$1/v = \xi_0 v^0 + \xi_1 v + \dots + \xi_{n-1} v^{n-1},$$

with $\xi_k \in R$ $(k = 0,1, \ldots, n-1)$ to be determined. Multiplying by v and taking into account (2_2), we get

$$v^0 = \xi_0 v + \xi_1 v^2 + \cdots - \xi_{n-1}(\alpha_1 v^{n-1} + \alpha_2 v^{n-2} + \cdots + \alpha_{n-1} v + \alpha_n v^0)$$

and finaly

$$\frac{1}{v} = -\frac{1}{\alpha_n}(\alpha_{n-1} v^0 + \alpha_{n-2} v + \cdots + \alpha_1 v^{n-2} + v^{n-1}). \tag{4}$$

3. Consider now the hypercomplex variable

$$z = x v^0 + y v \tag{5}$$

and the function

$$f(z) = \sum_{i=1}^{n} f_i(x, y)\, v^{i-1}. \tag{6}$$

We introduce the derivative in the elementary way used frequently in the theory of complex functions

$$f'(z) = \lim_{\Delta z \to 0} \frac{\Delta f}{\Delta z} = \lim_{\Delta z \to 0} \sum_{i-1}^{n} \frac{\Delta f_i(x, y)\, v^{i-1}}{\Delta x v^0 + \Delta y\, v} \tag{7}$$

and impose that $f'(z)$ be independent of the „direction" in which z changes. We conclude that

$$\sum_{i=1}^{n} \frac{\partial f_i}{\partial x} v^{i-1} \qquad \text{and} \qquad \sum_{i=1}^{n} \frac{\partial f_i}{\partial y} v^{i-2}$$

must be equal and we have

$$\frac{\partial f_1}{\partial x} v^0 + \frac{\partial f_2}{\partial x} v + \cdots + \frac{\partial f_n}{\partial x} v^{n-1} = \frac{1}{v} \frac{\partial f_1}{\partial y} + \frac{\partial f_2}{\partial y} v^0 + \cdots + \frac{\partial f_n}{\partial y} v^{n-2}.$$

Using (4) it results a system of n conditions

$$\left\{ \begin{aligned} \frac{\partial f_1}{\partial x} &= -\frac{\alpha_{n-1}}{\alpha_n} \frac{\partial f_1}{\partial y} + \frac{\partial f_2}{\partial y}, \\ \frac{\partial f_2}{\partial x} &= -\frac{\alpha_{n-2}}{\alpha_n} \frac{\partial f_1}{\partial y} + \frac{\partial f_3}{\partial y}, \\ &\cdots\cdots\cdots\cdots \\ \frac{\partial f_n}{\partial x} &= -\frac{1}{\alpha_n} \frac{\partial f_1}{\partial y}. \end{aligned} \right. \tag{8}$$

It can be written as

$$(9)\qquad \begin{cases} \dfrac{\partial f_k}{\partial x} = -\dfrac{\alpha_{n-k}}{\alpha_n}\,\dfrac{\partial f_1}{\partial y} + \dfrac{\partial f_{k+1}}{\partial y}, & (k = 1, 2, \ldots, n). \\ \text{with } \alpha_0 = 1, \quad f_{n+1} = 0. \end{cases}$$

We obtain easily that function $f_1(x, y)$ (i. e. the „real part" of (6)) verifies the partial differential equation

$$(10)\qquad \begin{aligned} 0 = \alpha_n \frac{\partial^n f_1}{\partial x^n} + \alpha_{n-1} \frac{\partial^n f_1}{\partial x^{n-1}\partial y} + \cdots + \alpha_2 \frac{\partial^n f_1}{\partial x^2 \partial y^{n-2}} + \\ + \alpha_1 \frac{\partial^n f_1}{\partial x\, \partial y^{n-1}} + \frac{\partial^n f_1}{\partial y^n}. \end{aligned}$$

In means that to any equation (10), with α constants, can be attached an algebra (2_1), in order to study the solutions by means of hypercomplex functions (6) defined on this algebra. This interesting fact will be explored in the future.

Remark. If (10) reduces to the biharmonic equation

$$\frac{\partial^4 f}{\partial x^4} + 2\,\frac{\partial^4 f}{\partial x^2 \partial y^2} + \frac{\partial f^4}{\partial y} = 0,$$

algebra (2_1) is a well known one, namely Sobrero algebra, initially introduced [2] with the aim to study by means of a function like (6), this biharmonic equation.

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ASUPRA UNOR FUNCŢII DE VARIABILĂ HIPERCOMPLEXĂ

(Rezumat)

Se demonstrează că soluţia ecuaţiei (10) este „partea reală" a funcţiei (6) de variabila complexă (5) pe algebra cu baza ciclică (2_2).

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SURFACES ALGÉBRIQUES AYANT UNE COURBE CANONIQUE ISOLÉE

PAR

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Nous avons démontré qu'une surface algébrique de genres $p_a = p_g = 0$ ayant un système bicanonique irréductible, était l'image d'une involution du second ordre, privée de points unis, appartenant à une surface algébrique ayant une courbe canonique isolée [1]. Nous avons ensuite donné une autre démonstration de ce théorème par une voie plus simple [2]. Dans cette Note nous reprenons cette démonstration en simplifiant encore certains points et nous considérons les hyperquadriques circonscrites aux surfaces en question.

Après avoir rappelé les propriétés des surfaces de genres $p_a = p_g = 0$, $P_2 > 2$ qui nous sont utiles, nous construisons la surface contenant l'involution d'ordre deux. Il est probable que la considération des hyperquadriques circonscrites aux surfaces considérées conduira à une démonstration géométrique d'un théorème que nous avons établi autrefois en utilisant une formule de Picard, à savoir que le bigenre de la surface de genres $p_a = p_g = 0$ est au plus égal à dix [3].

1. Soit Φ une surface de genres $p_a = p_g = 0$, $P_2 = p^{(1)}$ supérieur à deux, dont le système bicanonique est irréductible. Pour abréger, nous poserons $P_2 = \pi$. Nous avons démontré qu'il existe sur la surface Φ une courbe algébrique Γ isolée, de genre π et de degré virtuel $\pi - 1$. Le système bicanonique est $|2\Gamma|$, bien que la courbe Γ ne soit pas une courbe canonique. Le système $|2\Gamma|$ découpe sur la courbe Γ une série paracanonique.

Les systèmes pluricanoniques sont

$$|\Gamma_{2n-1}| = |(2n-3)\Gamma + \Gamma'|, \quad |\Gamma_{2n}| = |2n\Gamma|.$$

Le genre et le degré du système $|\Gamma_n|$ sont respectivement $\frac{1}{2}n(n+1)(\pi-1)+1$, $n^2(\pi-1)$. Le n-genre est donné par $P_n = \frac{1}{2}n(n-1)(\pi-1)+1$.

On a $2\Gamma' \equiv 4\Gamma$ et le diviseur de Severi de la surface Φ est $\sigma = 2$.

Il existe sur Φ des systèmes linéaires

$$|\bar{\Gamma}_{2n-1}| = |(2n-1)\Gamma|, \quad |\bar{\Gamma}_{2n}| = |(2n-2)\Gamma + \Gamma'|$$

et on a $2\Gamma_{2n-1} \equiv 2\bar{\Gamma}_{2n-1}$, $2\Gamma_{2n} \equiv 2\bar{\Gamma}_{2n}$.

Les systèmes $|\Gamma_{2n-1}|$ et $|\bar{\Gamma}_{2n-1}|$ d'une part, les systèmes $|\Gamma_{2n}|$ et $|\bar{\Gamma}_{2n}|$ d'autre part, ont les mêmes caractères (degré, genre, dimension).

Pour abréger, nous poserons $\nu = \pi - 1$.

2. Considérons dans un espace Σ à $2n(2n-1)\nu + 1$ dimensions, deux espace σ, $\bar{\sigma}$ à $n(2n-1)\nu$ dimensions, ne se rencontrant pas.

Dans le premier σ de ces espaces, nous considérons une surface Φ, d'ordre $4n^2\nu$, dont les sections hyperplanes sont les courbes $2n$-canoniques Γ_{2n} et dans le second, $\bar{\sigma}$, une transformée birationnelle de Φ dont les sections hyperplanes sont les courbes $\bar{\Gamma}_{2n}$. Soit $\bar{\Phi}$ cette surface.

Nous désignerons par le même symbole les courbes qui se correspondent sur les surfaces Φ et $\bar{\Phi}$.

Appelons T la correspondance birationnelle qui fait passer de Φ à $\bar{\Phi}$ et soit V_3 la variété lieu des droites joignant les points homologues de ces surfaces.

Considérons une courbe Γ_{2n} de Φ. Un hyperplan de Σ passant par cette courbe rencontre la courbe homologue de la surface $\bar{\Phi}$ en $4n^2\nu$ points. L'hyperplan contient donc $4n^2\nu$ droites de la variété V_2 lieu des droites joignant les points homologues de deux courbes Γ_{2n}. Un hyperplan passant par σ coupe V_2 suivant ces droites et passe simplement par la courbe Γ_{2n}. La variété V_2 est donc d'ordre $8n^2\nu$. Observons que la variété V_2 se trouve dans un hyperplan passant par $\bar{\sigma}$.

De même, la variété $\bar{V}_2$ lieu des droites s'appuyant sur deux courbes $\bar{\Gamma}_{2n}$ homologues des surfaces Φ, $\bar{\Phi}$ est d'ordre $8n^2\nu$ et est située dans un hyperplan passant par σ.

Un hyperplan passant par σ rencontre la variété V_3 suivant une variété $\bar{V}_2$ et passe simplement par Φ. La variété V_3 est donc d'ordre $12n^2$.

3. Les hyperquadriques de σ découpent sur la surface Φ des courbes du système $|4\Gamma|$ et $|4\bar{\Gamma}|$. De même, les hyperquadriques de l'espace $\bar{\sigma}$ découpent sur Φ des courbes des systèmes $|4\bar{\Gamma}|$ et $|4\Gamma|$. On peut donc imaginer une hyperquadrique $\varphi_2 = 0$ de σ découpant sur Φ une courbe 4Γ à laquelle correspond sur $\bar{\Phi}$ une courbe du système $|4\Gamma|$ découpée par une

hyperquadrique $\bar{\varphi}_2 = 0$ de σ. Les droites s'appuyant sur ces courbes en des points homologues dans T engendrent une réglée V_2' d'ordre $16n^2\nu$.

Cela étant, considérons dans Σ l'hyperquadrique $\varphi_2 + \bar{\varphi}_2 = 0$. Cette hyperquadrique coupe la variété V_3 suivant la variété V_2' d'ordre $16n^2\nu$ et suivant une surface F d'ordfe $8n^2\nu$.

Désignons par H l'homographie biaxiale harmonique de Σ ayant pour axes les espaces σ et $\bar{\sigma}$. Cette homographie transforme la surface F en elle-même et détermine sur cette surface une involution I d'ordre deux. Un couple de points de cette involution détermine une droite rencontrant les surfaces Φ et $\bar{\Phi}$ en des points homologues dans T.

Les hyperplans passant par les espaces σ ou $\bar{\sigma}$ déterminent sur F les courbes qui correspondent aux courbes $\bar{\Gamma}_{2n}$ de $\bar{\Phi}$ et Γ_{2n} de Φ. Ces courbes sont en général irréductibles, donc la surface F est irréductible.

Si l'involution I possédait une courbe unie, les droites de V_3 passant par les points de cette courbe appartiendraient à la surface F, qui serait réductible. Donc la surface F contient une involution I ayant au plus un nombre fini de points unis.

Appelons C_{2n} les courbes qui correspondent sur F aux courbes Γ_{2n} et $\bar{C}_{2n}$ celles qui correspondent aux courbes $\bar{\Gamma}_{2n}$. Ces courbes appartinent au système $|C_{2n}|$ des sections hyperplanes de F.

La surface Φ est une image de l'involution I. Considérons sur Φ une courbe Γ_{2n-1} ne passant par auncun des points de diramation éventuels de Φ. Il lui correspond sur la surface F une courbe C_{2n-1}. Les courbes C_{2n} découpent sur la courbe C_{2n-1} la série canonique complète, car la courbe C_{2n-1} est de genre $2n(2n-1)\nu + 1$ et la dimension de $|C_{2n}|$ est égale à $2n(2n-1)\nu+1$. Il en résulte, d'après un théorème cla si que de Castelnuovo, que la surface F est régulière $(p_a = p_g)$.

De plus, il existe une courbe C_{2n} contenant comme partie une courbe C_{2n-1} et la surface F possède une courbe canonique C_1, unique. On a donc, pour la surface F, $p_a = p_g = 1$.

Supposons maintenant que l'involution I possède x points unis. Entre le genre arithmétique $p_a = 1$ de F et celui, $p_a' = 0$ de Φ, on a la relation [4]

$$12(p_a + 1) = 2.12(p_a' + 1) - 4x,$$

d'où $x = 0$.

La surface F est régulière, possède une courbe canonique isolée et contient une involution du second ordre privée de points unis.

4. Le raisonnement précédent montre que le système canonique d'une courbe C_{2n-1} transformée d'une courbe $\bar{\Gamma}_{2n-1}$ est découpé par les courbes $\bar{C}_{2n} = C_{2n}$. On a donc

$$|C_{2n-1}| = |C_{2n}|, \qquad |C_{2n-1}'| = |C_{2n}|,$$

d'où

$$C_{2n} \equiv C_{2n-1} + C_1, \qquad C_{2n} \equiv \bar{C}_{2n-1} + C_1.$$

On en déduit que les courbes C_{2n-1} et $\bar{C}_{2n-1}$ appartiennent à un même système linéaire $|C_{2n-1}|$.

Soit m un nombre positif inférieur à $2n$. Désignons par C_m, $\bar{C}_m$ les courbes qui correspondent sur F aux courbes Γ_m, $\bar{\Gamma}_m$ des surfaces Φ, $\bar{\Phi}$. Le raisonnement fait plus haut montre que si les systèmes $|C_{m+1}|$, $|\bar{C}_{m+1}|$ coïncident, ils sont les adjoints aux courbes C_m et $\bar{C}_m$. On a donc $C_{m+1} \equiv \equiv C_m + C_1$, $C_{m+1} \equiv \bar{C}_m + C_1$, d'où l'on conclut que les systèmes $|C_m|$ et $|\bar{C}_m|$ coïncident en un même système $|C_m|$.

La propriété a été démontrée pour $m = 2n - 1$, donc elle est vraie quelque soit m.

5. En particulier on a pour $m = 1$, que le système $|C_2|$ est l'adjoint à la courbe C qui correspond sur F à la courbe Γ des surfaces Φ et $\bar{\Phi}$. On a par suite $C_2 - C \equiv C \equiv C_1$. La courbe Γ est unique sur les surfaces Φ et $\bar{\Phi}$ et la courbe C_1 est unique sur la surface F. Elle est située sur la surface lieu des droites joignant les points homologues dans T des courbes Γ et $\bar{\Gamma}$ situées sur les surfaces Φ et $\bar{\Phi}$.

Les genre et degré des systèmes $|\bar{\Gamma}_m|$, $|\Gamma_m|$ sont respectivement $\frac{1}{2}m(m+1)\nu + 1$, $m^2\nu$.

Par conséquent le genre et le degré du système $|C_m|$ sont respectivement $m(m+1)\nu + 1$, $2m^2\nu$.

La dimension P'_m du système $|C_m|$ est donnée par $P'_m - 1 = m(m - -1)\nu + 1 = 2P_m + 1$.

Si l'on désigne par π_{m-1} le genre des courbes C_{m-1}, on a $\pi_{m-1} = = m(m-1)\nu + 1 = P'_m - 1$.

Sur la surface Φ, les courbes Γ_{2n} découpent sur la courbe Γ une série linéaire d'ordre $2n\nu$. La courbe Γ étant de genre $\pi = \nu + 1$, cette série a la dimension $2n\nu - \pi = (2n-1)\nu - 1$. Il en résulte que la courbe Γ appartient à un espace à $(2n-1)\nu - 1$ dimensions. Par conséquent la courbe C_1 appartient à un espace à $2(2n-1)\nu - 1$ dimensions si, comme nous le supposerons désormais, on a $n > 1$.

Observons qu'à une courbe Γ de Φ, ou $\bar{\Gamma}$ de $\bar{\Phi}$, correspond sur F une courbe C et qu'à une courbe Γ' correspond une courbe $2C$. Le système m-canonique de la surface F est donc $|C_m| = |mC_1|$.

6. Dans l'espace σ, il existe un hyperplan ξ passant par la courbe Γ et ayant le long de cette courbe un contact d'ordre $2n - 1$ avec la surface Φ. En d'autre termes, l'hyperplan ξ rencontre la surface Φ suivant la courbe Γ comptée $2n$ fois.

Les hyperquadriques de l'espace σ linéairement indépendantes sont au nombre de $\frac{1}{2}[n(2n-1)\nu + 2][n(2n-1)\nu + 1]$.

Elles rencontrent la surface Φ suivant les courbes du système $|4\Gamma|$, ou suivant les courbes découpées par les hyperplans distincts de ξ quand l'hyperquadrique contient cet hyperplan, ou contiennent la surface Φ. Il en résulte que les hyperquadriques linéairement indépendantes contenant la surface Φ sont au nombre de

$$\frac{1}{2}[n^2(2n-1)^2\nu^2+3n(2n-1)\nu+2]-[2n(4n-1)\nu+1]-$$

$$-n(2n-1)\nu=\frac{1}{2}[n^2(2n-1)^2\nu^2-n(14n-3)\nu].$$

Sur la surface Φ, la courbe Γ est d'ordre $2n\nu$ et de genre $\nu+1$. Nous avons supposé $n>1$. Dans cette condition, nous avons vu que la courbe Γ appartient à un espace σ_1 de dimension $(2n-1)\nu-1$.

Les hyperquadriques de σ_1 linéairement indépendantes passant par la courbe Γ sont au nombre de $\frac{1}{2}[(2n-1)^2\nu^2-(6n-1)\nu]$.

Les hyperquadriques de σ contenant Φ découpent sur σ_1 ces hyperquadriques ou contiennent l'espace σ_1. Il existe donc

$$\frac{1}{2}[(n^2-1)(2n-1)^2\nu^2-2n(4n-1)\nu]$$

hyperquadriques linéairement indépendantes contenant à la fois la surface Φ et l'espace σ_1 de la courbe Γ.

Les raisonnements précédents peuvent être répétés pour la surface $\bar{\Phi}$ de $\bar{\sigma}$. Il existe dans $\bar{\sigma}$ un hyperplan $\bar{\xi}$ ayant un contact d'ordre $2n-1$ avec la surface $\bar{\Phi}$ le long de la courbe $\bar{\Gamma}$ que T fait corresponder à Γ sur la surface $\bar{\Phi}$. La courbe $\bar{\Gamma}$ appartient à un expace $\bar{\sigma}_1$ à $(2n-1)\nu-1$ dimensions.

7. Dans l'espace Σ, il existe un hyperplan ξ_0 contenant la courbe C_1 ayant un contact d'ordre $2n-1$ avec la surface F le long de cette courbe.

Les hyperquadriques de l'espace Σ linéairement indépendantes sont au nombre de $2n^2(2n-1)^2\nu^2+5(2n-1)\nu+3$. Ces hyperquadriques découpent sur la surface F les courbes du système $|4\Gamma|$ ou les sections de F par les hyperplans distincts de ξ_0 lorsque l'hyperquadrique contient cet hyperplan, ou contiennent la surface F. Les hyperquadriques de Σ qui contiennent la surface F, linéairement indépendantes, sont donc au nombre de $2n^2(2n-1)^2\nu^2-n(10n-1)\nu$.

Sur une courbe C_1, les sections hyperplanes de F découpent une série linéaire d'ordre $4n\nu$ non spéciale puisque $n>1$. Le genre de la courbe C_1 étant $2\nu+1$, cette série a la dimension $2(2n-1)\nu-1$. La courbe C_1 appartient donc à un espace Σ_1 à $2(2n-1)\nu-1$ dimensions. Cet espace contient les espaces σ_1 et $\bar{\sigma}_1$ et est complètement déterminé par eux.

Les hyperquadriques de l'espace Σ_1 linéairement indépendantes sont au nombre de $2(2n-1)^2\nu^2+(2n-1)\nu$. Elles découpent sur la courbe

C_1 une série non spéciale d'ordre $8n\nu$ et de dimension $2(2n-1)\nu-1$. Il y a donc $2(2n-1)^2\nu^2-(6n-1)\nu$ hyperquadriques de Σ_1 passant par C_1.

Les hyperquadriques de Σ contenant F rencontrent l'espace Σ_1 suivant les hyperquadriques contenant C_1 ou contiennent Σ_1. Il y a donc $2(n^2-1)(2n-1)^2$ hyperquadriques de Σ linéairement indépendantes contenant à la fois la surface F et l'espace Σ_1 .

8. La courbe C_1 se trouve sur la surface F_1 lieu des droites s'appuyant en des points homologues dans T sur les courbes Γ et $\bar{\Gamma}$. Cette surface a l'ordre $4n^2\nu$. Elle est située dans l'espace Σ_1 à $2(2n-1)\nu-1$ dimensions.

Désignons par $\psi_2=0$ l'équation d'une hyperquadrique de σ_1 passant par la courbe Γ. Dans Σ_1, cette équation représente un cone quadratique lieu des droites s'appuyant sur Γ et sur $\bar{\sigma}$ et par conséquent contenant la surface F_1.

De même, si $\bar{\psi}_2=0$ est l'équation d'une hyperquadrique de $\bar{\sigma}_1$ passant par le courbe $\bar{\Gamma}$, dans Σ_1 cette équation représente un cône quadratique passant par la surface F_1.

Il en résulte que dans Σ_1, l'équation $\psi_2+\bar{\psi}_2=0$ représente une hyperquadrique passant par la surface F_1. On obtient ainsi $\frac{1}{4}[(2n-1)^2\nu^2-(6n-1)\nu]^2$ hyperquadriques linéairement indépendantes contenant la surface F_1. Ce nombre est inférieur à celui des hyperquadriques de Σ contenant la surface F.

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SUPRAFEŢE ALGEBRICE AVÎND O CURBĂ CANONICĂ IZOLATĂ

(Rezumat)

Continuînd seria de lucrări privitoare la studiul unor suprafeţe algebrice [1], [2], se stabilesc noi rezultate şi se consideră între altele hipercuadricele circumscrise suprafeţelor studiate.

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BIANCHI'S IDENTITIES IN A NON SYMMETRIC CONNECTION SPACE

BY

CĂTĂLINA NIŢESCU

1. We consider a n — dimensional space A_n with non-symmetric connection Γ^i_{jk} ($\Gamma^i_{jk} \neq \Gamma^i_{kj}$) which is based on non-symmetric tensor $g_{ij} \neq g_{ji}$.

The connection Γ^i_{jk} can be written as [1]:

$$\Gamma^i_{rs} = S^i_{rs} + \frac{1}{2} T^i_{rs} , \tag{1}$$

where S^i_{rs} and $\frac{1}{2} T^i_{rs}$ are the symmetric and skew-symmetric parts of Γ^i_{jk}. Let us introduce [2] a conection $\widetilde{\Gamma}^i_{jk}$ defined by

$$\widetilde{\Gamma}^i_{rs} = S^i_{rs} - \frac{1}{2} T^i_{rs} . \tag{2}$$

Then from (1) and (2) we have:

$$\widetilde{\Gamma}^i_{jk} = \Gamma^i_{kj} . \tag{3}$$

If semicolon (;) followed by a pair of indices denotes covariant derivative, then the covariant derivative of a contravariant vector u^i will be defined [2] in two ways as follows:

$$\overset{i}{u}{}^{+}_{;j} = u^i_{,j} + u^k \Gamma^i_{kj} , \tag{4}$$

$$\overset{i}{u}{}^{-}_{;j} = u^i_{,j} + u^k \widetilde{\Gamma}^i_{kj} . \tag{5}$$

The equation (5) can also be written as:

$$\overset{i}{u}{}^{-}_{;j} = u^i_{,j} + u^k \Gamma^i_{jk},$$

where comma (,) followed by a pair of indices denotes partial derivative, a positive (+) sign below a pair of indices followed by a semicolon indicates that the covariant derivative has been formed with respect to the connection Γ^i_{jk} as far as that index is concerned and a negative sign beleow an index followed by semicolon indicates that the covariant derivative has been formed with respect to the connection $\widetilde{\Gamma}^i_{jk}$ concerning that index.

Similarly can be extended the definition of covariant derivative of a tensor of any order. This duality in the nature of covariant derivative introduces two curvature tensors, given by :

$$R^i_{jkl} = \Gamma^i_{jk,l} - \Gamma^i_{jl,k} + \Gamma^i_{\beta l}\,\Gamma^\beta_{jk} - \Gamma^i_{\beta k}\,\Gamma^\beta_{jl}\,,$$

$$\widetilde{R}^i_{jkl} = \widetilde{\Gamma}^i_{jk,l} - \widetilde{\Gamma}^i_{jl,k} + \widetilde{\Gamma}^i_{\beta l}\widetilde{\Gamma}^\beta_{jk} - \widetilde{\Gamma}^l_{\beta k}\widetilde{\Gamma}^\beta_{jl} = \Gamma^i_{kj,l} - \Gamma^i_{lj,k} + \Gamma^i_{l\beta}\,\Gamma^\beta_{kj} - \Gamma^i_{k\beta}\,\Gamma^\beta_{lj}\,.$$

2. B i a n c h i 's identities with respect to symmetric connection L^i_{jk} are given [3] by

$$B^i_{jkl/m} + B^i_{jlm/k} + B^i_{jmk/l} = 0,$$

where B^i_{jkl} is the curvature tensor in A_n and (/) denotes the covariant derivative with regard to the connection L^i_{jk}. Obviously these relations will not be satisfied if the connection L^i_{jk} is a non-symmetric one.

For the considered connection, Γ^i_{jk} the sum

$$\text{(a)} \qquad R^h_{ijk;l} + R^h_{ikl;j} + R^h_{ilj;k}$$

is not equal with zero, allthough the second order partial derivatives of Γ^i_{jk} can be canceled

$$\begin{aligned} R^h_{ijk;l} + R^h_{ikl;j} + R^h_{ilj;k} = R^h_{ijk,l} + R^h_{ikl,j} + R^h_{ilj,k} + E = (\Gamma^h_{ij,kl} - \Gamma^h_{ik,jl} + \\ + \Gamma^h_{\beta k,l}\,\Gamma^\beta_{ij} + \Gamma^h_{\beta k}\,\Gamma^\beta_{ij,l} - \Gamma^h_{\beta j,l}\,\Gamma^\beta_{ik} - \Gamma^h_{\beta j}\,\Gamma^\beta_{ik,l}) + (\Gamma^h_{ik,lj} - \Gamma^h_{il,kj} + \\ + \Gamma^h_{\beta l,j}\,\Gamma^\beta_{ik} - \Gamma^h_{\beta k,j}\,\Gamma^\beta_{il} + \Gamma^h_{\beta l}\,\Gamma^\beta_{ik,j} - \Gamma^h_{\beta k}\,\Gamma^\beta_{il,j}) + (\Gamma^h_{il,jk} - \Gamma^h_{ij,lk} + \\ + \Gamma^h_{\beta j,k}\,\Gamma^\beta_{il} - \Gamma^h_{\beta l,k}\,\Gamma^\beta_{ij} + \Gamma^h_{\beta j}\,\Gamma^\beta_{il,k} - \Gamma^h_{\beta l}\,\Gamma^\beta_{ij,k}) + E. \end{aligned}$$

Finally from the partial derivatives of curvature tensor will remain twelve terms which are to be canceled with the terms of E. We need to choose the signs under indices such that the covariant derivatives to be taken with respect to connection

$$\Gamma^i_{jk} \qquad \text{or} \qquad \widetilde{\Gamma}^i_{jk} = \Gamma^i_{kj}\,.$$

Our aim is to cancel all terms in the right side of (a) and so to obtain B i a n c h i 's identities.

Let us prove that :

$$(\mathrm{I}_1) \qquad \overset{h}{\underset{+}{R}}{}^{+}_{ijk;l} + \overset{h}{\underset{+}{R}}{}^{+}_{ikl;j} + \overset{h}{\underset{+}{R}}{}^{+}_{ilj;k} = 0\,.$$

By a convenient choosing of signs, E will become :

$$E = (R^{\alpha}_{ijk}\,\Gamma^{h}_{\alpha l} - R^{h}_{\beta jk}\,\Gamma^{\beta}_{il}) + (R^{\alpha}_{ikl}\,\Gamma^{h}_{\alpha j} - R^{h}_{\beta kl}\,\Gamma^{\beta}_{ij}) + (R^{\alpha}_{ilj}\,\Gamma^{h}_{\alpha k} - R^{h}_{\beta lj}\,\Gamma^{\beta}_{ik}) .$$

If (I_1), is developed all terms cancel :

$$R^{h}_{ijk,\,l} = \Gamma^{h}_{ij,\,kl} - \Gamma^{h}_{ik,\,jl} + \Gamma^{h}_{\beta k,\,l}\,\Gamma^{\beta}_{ij} + \Gamma^{h}_{\beta k}\,\Gamma^{\beta}_{ij,\,l} - \Gamma^{h}_{\beta j,\,l}\,\Gamma^{\beta}_{ik} - \Gamma^{h}_{\beta j}\,\Gamma^{\beta}_{ik,\,l} , \tag{6}$$

$$R^{h}_{ikl,\,j} = \Gamma^{h}_{ik,\,lj} - \Gamma^{h}_{il,kj} + \Gamma^{h}_{\beta l,\,j}\,\Gamma^{\beta}_{ik} - \Gamma^{h}_{\beta k,\,j}\,\Gamma^{\beta}_{il} + \Gamma^{h}_{\beta l}\,\Gamma^{\beta}_{ik,\,j} - \Gamma^{h}_{\beta k}\,\Gamma^{\beta}_{il,\,j} , \tag{7}$$

$$R^{h}_{ilj,\,k} = \Gamma^{h}_{il,\,jk} - \Gamma^{h}_{ij,\,lk} + \Gamma^{h}_{\beta j,\,k}\,\Gamma^{\beta}_{il} - \Gamma^{h}_{\beta l,\,k}\,\Gamma^{\beta}_{ij} + \Gamma^{h}_{\beta j}\,\Gamma^{\beta}_{il,\,k} - \Gamma^{h}_{\beta l}\,\Gamma^{\beta}_{ij,\,k} , \tag{8}$$

$$\begin{aligned} R^{\alpha}_{ijk}\,\Gamma^{h}_{\alpha l} - R^{h}_{\beta jk}\,\Gamma^{\beta}_{il} = {} & (\Gamma^{\alpha}_{ij,\,k} - \Gamma^{\alpha}_{ik,\,j} + \Gamma^{\alpha}_{\gamma k}\,\Gamma^{\gamma}_{ij} - \Gamma^{\alpha}_{\delta j}\,\Gamma^{\delta}_{ik})\,\Gamma^{h}_{\alpha l} - \\ & - (\Gamma^{h}_{\beta j,\,k} - \Gamma^{h}_{\beta k,\,j} + \Gamma^{h}_{\gamma k}\,\Gamma^{\gamma}_{\beta j} - \Gamma^{h}_{\delta j}\,\Gamma^{\delta}_{\beta k})\,\Gamma^{\beta}_{il} , \end{aligned} \tag{9}$$

$$\begin{aligned} R^{\alpha}_{ikl}\,\Gamma^{h}_{\alpha j} - R^{h}_{\beta kl}\,\Gamma^{\beta}_{ij} = {} & (\Gamma^{\alpha}_{ik,\,l} - \Gamma^{\alpha}_{il,\,k} + \Gamma^{\alpha}_{\gamma l}\,\Gamma^{\gamma}_{ik} - \Gamma^{\alpha}_{\delta k}\,\Gamma^{\delta}_{il})\,\Gamma^{h}_{\alpha j} - \\ & - (\Gamma^{h}_{\beta k,\,l} - \Gamma^{h}_{\beta l,\,k} + \Gamma^{h}_{\gamma l}\,\Gamma^{\gamma}_{\beta k} - \Gamma^{h}_{\delta k}\,\Gamma^{\delta}_{\beta l})\,\Gamma^{\beta}_{ij} , \end{aligned} \tag{10}$$

$$\begin{aligned} R^{\alpha}_{ilj}\,\Gamma^{h}_{\alpha k} - R^{h}_{\beta lj}\,\Gamma^{\beta}_{ik} = {} & (\Gamma^{\alpha}_{il,\,j} - \Gamma^{\alpha}_{ij,\,l} + \Gamma^{\alpha}_{\gamma j}\,\Gamma^{\gamma}_{il} - \Gamma^{\alpha}_{\delta l}\,\Gamma^{\delta}_{ij})\,\Gamma^{h}_{\alpha k} - \\ & - (\Gamma^{h}_{\beta l,\,j} - \Gamma^{h}_{\beta j,\,l} + \Gamma^{h}_{\gamma j}\,\Gamma^{\gamma}_{\beta l} - \Gamma^{h}_{\delta l}\,\Gamma^{\delta}_{\alpha j})\,\Gamma^{\beta}_{ik} . \end{aligned} \tag{11}$$

Adding relations, (1)...(6), we obtain :

$$\underset{+}{\overset{h}{R}}{}^{+}_{ijk\,;\,l} + \underset{+}{\overset{h}{R}}{}^{+}_{ikl;\,j} + \underset{+}{\overset{h}{R}}{}^{+}_{ilj;\,k} = 0.$$

Similarly we can prove the existence of the following four relations

$$\underset{+\,+\,+}{\overset{h}{R}}{}^{+}_{ijk;\,l} + \underset{++-}{\overset{h}{R}}{}^{+}_{ikl\,;\,j} + \underset{+--}{\overset{h}{R}}{}^{+}_{ilj\,;\,k} = 0 , \tag{I_2}$$

$$\underset{+-+}{\overset{h}{R}}{}^{+}_{ijk\,;\,l} + \underset{+-+}{\overset{h}{R}}{}^{+}_{ikl\,;\,j} + \underset{+-+}{\overset{h}{R}}{}^{+}_{ilj\,;\,k} = 0, \tag{I_3}$$

$$\underset{++-}{\overset{h}{R}}{}^{+}_{ijk;\,l} + \underset{++-}{\overset{k}{R}}{}^{+}_{ihl\,;\,j} + \underset{++-}{\overset{h}{R}}{}^{+}_{ilj\,;\,h} = 0, \tag{I_4}$$

$$\underset{+}{\overset{h}{R}}{}^{+}_{ijk;\,l} + \underset{+--}{\overset{h}{R}}{}^{+}_{ikl\,;\,j} + \underset{+-+}{\overset{h}{R}}{}^{+}_{ilj;\,k} = 0. \tag{I_5}$$

In these identities the contractions are possible if the indices have the sign plus under them.

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IDENTITĂŢI BIANCHI ÎNTR-UN SPAŢIU CU CONEXIUNE NESIMETRICĂ

(Rezumat)

Pornind de la identităţile lui Bianchi într-un spaţiu cu conexiune simetrică se dă o generalizare a acestora într-un spaţiu cu conexiune nesimetrică.

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ON A NEW STABILITY CRITERION [1])

BY

ARTHUR L. SCHOENSTADT

1. Introduction

Most classical results (see Cesari [1]) on the stability of

$$x'(t) = A(t) + f(x, t), \quad t > t_0, \tag{1}$$

where $A(t)$ is continuous and,

$$f(x, t) \mid \leqslant c(t) \mid x \mid \quad \text{and} \quad \int_{t_0}^{\infty} c(t)\, dt > \infty, \tag{1'}$$

are stated in terms of criteria on the fundamental matrix of

$$x'(t) = A(t)\, x(t), \quad t < t_0, \tag{2}$$

i. e., the matrix solution of

$$\begin{aligned} \Phi'(t) &= A(t)\, \Phi(t), \\ \Phi(t_0) &= I. \end{aligned} \tag{3}$$

The most commonly shown results ([1], [2]) occur when $\Phi(t)$ satisfies one or more of the following conditions (where $\| . \|$ stands for a matrix norm, and $| . |$ for a vector norm such that $\| AB \| \leqslant \| A \| \cdot \| B \|$, and $\| Ax \| \leqslant \leqslant \| A \| \; | x |$:

a. *Asymptotic Stability* — $\lim_{t \to \infty} \| \Phi(t) \| = 0$,

b. *Uniform Stability* — $\| \Phi(t)\, \Phi^{-1}(s) \| \leqslant M$, all $t_0 \leqslant s \leqslant t \leqslant \infty$,

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[1]) The work accomplished on which this paper is based was supported partially by the Foundation Research Program, Naval Postgraduate School, which was funded by the Office of Naval Research, Arlington, Virginia.

c. *Strong* (Restrictive) *Stability* — $\| \Phi(t) \| \leqslant M$, and $\| \Phi^{-1}(t) \| \leqslant M$ for all $t_0 \leqslant t$.

Results for the above cases have been obtained when $f(x,t)$ is somehow small, and are covered in several texts besides [1].

2. Quasi-Strong Stability

We propose to investigate (1) under a condition we shall call quasi-strong stability. We shall show this condition is weaker than strong stability, and independent of uniform stability.

D e f i n i t i o n. The system (2) is said to be quasi-strongly stable if and only if $\Phi(t)$ is bounded on (t_0, ∞) and

$$\|\Phi(t)\| \, \| \Phi^{-1}(t) \| \leqslant M, \quad \text{for all} \quad t \geqslant t_0. \tag{4}$$

The following examples consider this criteria In these examples we use for the matrix norm $\| A \|^2 = \sum_{i,j} |a_{ij}|^2$. We also note that any continuously differentiable, non-singular matrix which reduces to the identity at some time t_0, can be viewed as the fundamental matrix for some system of type (2).

Example 1. (Quasi-Strong but not Strong). Let

$$\Phi(t) = e^{-t} \, \mathrm{I}.$$

Then $\Phi^{-1}(t) = \mathrm{e}^t \, \mathrm{I}$. But using our norm,

$$\| \Phi(t) \| = \mathrm{e}^{-t} \, n^{\frac{1}{2}} \quad \text{and} \quad \| \Phi^{-1}(t) \| = \mathrm{e}^t n^{\frac{1}{2}} \cdot$$

Thus $\Phi(t)$ is not strongly stable, but clearly quasi-strongly stable since

$$\| \Phi(t) \| \, \| \Phi^{-1}(t) \| = n.$$

Example 2. (Uniform but not Quasi-Strong). Let

$$\Phi(t) = \begin{bmatrix} \mathrm{e}^{it} & 0 & 0 \\ 0 & \mathrm{e}^{-it} & 0 \\ 0 & 0 & \mathrm{e}^{-t} \end{bmatrix}.$$

Then

$$\| \Phi(t) \| = (2 + \mathrm{e}^{-2t})^{\frac{1}{2}} \cdot$$

It is easily seen that $\| \Phi^{-1}(t) \| = (2 + \mathrm{e}^{2t})^{\frac{1}{2}}$, and the system is clearly not quasi-strongly stable. However, it is also easily seen that

$$\| \Phi(t) \| \, \|\Phi^{-1}(s) \| = (2 + \mathrm{e}^{-2(t-s)})^{\frac{1}{2}} \leqslant 3,$$

and so this system is uniformly stable.

Example **3**. (Quasi-Strong but not Uniform). Let

$$\Phi(t) = h(t)\, e^{-t} \begin{bmatrix} 1 & 0 \\ 0 & g(t) \end{bmatrix},$$

where

$$h(t) = 1 + (1 - e^{-t}) \cos t, \qquad g(t) = \left(1 + \frac{1}{2} \sin t\right).$$

Thus

$$\| \Phi(t) \| = h(t)\, e^{-t} (1 + g^2(t))^{\frac{1}{2}} \leqslant 3h(t) e^{-t}.$$

Now

$$\|\Phi^{-1}(t)\| = h^{-1}(t)\, e^{t} \begin{bmatrix} 1 & 0 \\ 0 & g^{-}(t) \end{bmatrix}$$

and so

$$\| \Phi^{-1}(t) \| = h^{-1}(t)\, e^{t} (1 + g^{-2}(t))^{\frac{1}{2}} \leqslant 3h^{-1}(t)\, e^{t},$$

and thus $\Phi(t)$ is clearly quasi-strongly stable. However, we can also show easily that

$$\| \Phi(t)\, \Phi^{-1}(s) \| = h(t)\, h^{-1}(s) e^{-(t-s)} (1 + g^2(t)\, g^{-2}(s))^{\frac{1}{2}} \geqslant$$
$$\geqslant h(t) h^{-1}(s) e^{-(t-s)}.$$

Now let $t_n = 2n\pi$, and $s_n = (2n - 1)\pi$. Thus

$$\| \Phi(t_n)\, \Phi^{-1}(s_n) \| \geqslant h(t_n)\, h^{-1}(s_n)\, e^{-\pi}\, (2 - e^{-s_n})\, e^{-\pi}$$

and clearly this is not bounded as $n \to \infty$. Thus $\Phi(t)$ is not uniformly stable.

We have thus demonstrated that quasi-strong stability is a weaker condition than strong stability, and is independent of uniform stability. It might be noted in passing that the system shown in example **3** is also asymptotically stable.

3. Stability and Asymptotic Stability under Quasi-Strong Stability

The system (1) is said to be stable if given $\varepsilon < 0$, there is a $\delta > 0$ such that $|x_0| < \delta$ implies $|x(t\,;\, x_0,\, t_0)| < \varepsilon$. We can thus show the following

Theorem. *Suppose that the fundamental matrix of $x' = A(t)y$ is quasi-strongly stable. Suppose also that*

$$|f(x,\, t)| \leqslant c(t)\, |x|, \qquad \text{where} \int_{t_0}^{\infty} c(t)\, dt < \infty.$$

Then the solution of

$$x' = A(t)x + f(x,\, t)$$

is stable.

Proof. Let $\Phi(t)$ denote the fundamental matrix of (2). Then by the assumptions above:

$$\| \Phi(t) \| \, \| \Phi^{-1}(t) \| \leqslant M.$$

It is well known that the solution of (1) also satisfies

$$x(t) = \Phi(t)x_0 + \Phi(t) \int_{t_0}^{t} \Phi^{-1}(s) f(x(s), s) \, ds.$$

Taking norms on both sides leads to

$$|x(t)| \leqslant \| \Phi(t) \| \, |x_0| + \| \Phi(t) \| \int_{t^0}^{t} \| \Phi^{-1}(s) \| \, |f(x(s), s)| \, ds \leqslant$$

$$\leqslant \| \Phi(t) \| \, |x_0| + \Phi(t) \| \int_{t_0}^{t} \| \Phi^{-1}(s) \| \, c(s) \, |x(s)| \, ds.$$

Clearly $\| \Phi(t) \| \neq 0$ at any finite t, and so

$$(|x(t)| \, / \, \| \Phi(t) \|) \leqslant |x_0| + \int_{t_0}^{t} \| \Phi^{-1}(s) \| \, c(s) \, |x(s)| \, ds.$$

But since (2) is quasi-strongly stable, we have that

$$\| \Phi^{-1}(s) \| \leqslant M / \| \Phi(s) \|.$$

Hence,

$$(| x(t) | \, / \, \|\Phi(t) \|) \leqslant | x_0 | + M \int_{t_0}^{t} c(s) \, (|x(s)| \, / \, \| \Phi(s) \|) \, ds.$$

However, this relation must hold for all $t \geqslant t_0$, and hence we are able to apply Gronwall's lemma ([1]), to yield that

$$(|x(t)| \, / \, \| \Phi(t) \|) \leqslant | x_0 | \exp \left(M \int_{t_0}^{t} c(s) \, ds \right) \leqslant | x_0 | \exp \left(M \int_{t_0}^{\infty} c(s) \, ds \right) = K \, |x_0|$$

for some real K, since the integral exists.

Cross multiplying yields that

$$|x(t)| \leqslant K \, |x_0| \cdot \| \Phi(t) \|,$$

and, since $\Phi(t)$ is bounded by some M_1, the follows, by picking $\sigma = \varepsilon/KM_1$.

We also recall that (1) is said to be globally asymptotically stable if $\lim_{t\to\infty} \| x(t\,;\, x_0,\, t_0) \| = 0$, for any x_0. From our theorem it is obvious that the following corollary is true:

Corollary. *Suppose that the fundamental matrix of $x' = A(t)$ is both quasistrongly stable and asymptotically stable. Suppose also that*

$$|f(x,\, t)| \leqslant c(t)\,|\,x\,|, \quad \textit{where} \quad \int_{t_0}^{\infty} c(t)\,\mathrm{d}t < \infty.$$

Then every solution of $x' = A(t)x + f(x,\, t)$, $t > t_0$, is globally asymptotically stable.

Proof: It is only necessary to note that under the assumptions of this corollary $|\,x(t\,;\, x_0,\, t_0)\,| \leqslant K\,|\,x_0\,|\,\|\Phi(t)\|$ where $\|\Phi(t)\|$ goes to zero at large t because of the asymptotic stability of the linear system.

4. Discussion

These results are parallel to results mentioned by Cesari, where his results are derived for uniform stability or strong stability. It should be noted, though, it was shown above that quasi-strong stability is independent of uniform stability. One result mentioned by Cesari which we were unable to show for quasi-strong stability was whether the stability derived is uniform for $t_1 \geqslant t_0$. It is our belief that this need not be so.

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ASUPRA UNUI CRITERIU DE STABILITATE

(Rezumat)

Majoritatea rezultatelor privind stabilitatea sistemelor descrise de cuaţia (1), cu condiţiile (1′), se exprimă în raport cu proprietăţile matricei fundamentale a lui $x' = A(t)x$. Lucrarea prezintă un nou criteriu privind matricea fundamentală care asigură stabilitatea sistemului şi indică legătura dintre acest criteriu şi cele utilizate în prezent.

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ON THE STABILITY OF THE STEADY-STATE SOLUTIONS OF TWO NEURAL MODELS [1] [2]

BY

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Dedicated to the 15th anniversary of the BIOLOGICAL COMPUTER LABORATORY of the University of Illinois

Introduction

„Living organisms perpetually compute on their sensory inputs complex abstracts, relations and decisions in order to determine the appropriate actions which will allow them to survive in a hostile and capricious environment called „Nature". They leave the naive task of computing sums, products, differences, fractions, square roots, log tables, Bessel functions, etc., to machines, for their nervous system is busy solving much more sophisticated computational problems, for instance, how to identify quickly and reliably desirable or dangerous entities in their environment, how to predict within a certain degree of confidence the future behaviour of these entities once identified and, ultimately, how to react, interact and relate to these entities once identified and understood. Today, if our times of the 'information explosion' in which we are ever more forced to process information generated and absorbed by man, a new discipline, called *Bionics*, has the task to find the answers as to how have living organisms solved this problem of gathering, processing and utilizing information which, when

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[1]) An excerpt of this paper was presented with a very favourable comment by M. Picone to the *Accademia Nazionale dei Lincei*.

[2]) This work was partly supported by the National Research Council of Canada under Grant NRC-A 4345 through the University of Alberta.

necessary, may be communicated via signs or symbols to other organisms of the same or different kind".

The Biological Computer Laboratory of the University of Illinois is successfuly addressing itself, just like many other similar laboratories throughout the world [3]), during the last fifteen years, to the question concerning exploration of the principles of computation in living organisms, the establishment of the structural and fonctional organization of such „biological computers", the utilization of this knowledge in the design and construction of cognitive systems, conversant systems, inductive inference machines, etc.

In this ample and important framework the authors further investigate in what follows the solutions of the neural equations considered in [15]. After studying the stability of the steady-state solutions to these equations in general, we discuss finite systems with more details, because their importance in laboratory work and computer simulations. A partial list of relevant references is given at the end of the present work.

In one of our next research work the numerical analysis of the solutions will be given and **FORTRAN** programs for the methods of variable step size and variable order of approximations will be exposed.

I. Statement of the Problem

The infinite systems of ordinary differential equations

$$\left(\tau \frac{d}{dt} + 1\right) \log \frac{x_i}{1 - x_i} = \gamma_i + \sum_{j=1}^{\infty} \beta_{ij} x_j \tag{I.1}$$

and

$$\left(\tau \frac{d}{dt} + a_i\right) x_i = \frac{1}{1 + \exp\left\{-\left[\gamma_i + \sum_{j=1}^{\infty} \beta_{ij} x_j\right]\right\}}, \tag{I.2}$$

$i = 1, 2, 3, \ldots$, have been proposed by J. D. Cowan and R. B. Stein (cf. [15]) respectively to describe the neural activities in a neural net. In (I.1) and (I.2), $x_i = x_i(t)$ denotes Cowan's „sensitivity" of the i-th cell, $0 < x_i < 1$, τ denotes the mean interspike interval, γ_i is the external input to the i-th cell, β_{ij} represents the strength of influence of the j-th cell to the i-th cell, $\beta_{ij} = 0$, and α_i represents the strength of self inhibition in the i-th cell.

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[3]) The authors will dedicate in the same *Bulletin* one of their future Notes to the appropriate anniversary of a set of researches pertaining to Mathematical Biology worked out by earlier Romanian scientists.

It has been shown in [15] that Eqs (I.1) and (I.2) admit unique solutions for $t > 0$ which satisfy the initial conditions

$$x_i(0) = x_{i,0} \qquad (0 < x_{i,0} < 1) \tag{I.3}$$

$i = 1, 2, 3, \ldots,$ and, these solutions are monotonic in sufficiently small intervals.

In the present work we first investigate the steady-state solutions of Eqs. (I.1) and (I.2) and their stability in the case in which neural nets contain infinitely many cells. Then, we consider finite nets with more details, because of their practical importance in laboratory work and computer simulations, particularly their stability properties.

II. Steady-State Solutions

In this section we examine the steady-state solutions of Eqs. (I.1) and (I.2). For this purpose, consider the infinite systems of equations

$$\log \frac{x_i}{1-x_i} = \gamma_i + \sum_{j=1}^{\infty} \beta_{ij} x_j ,$$

$$\alpha_i x_i = \frac{1}{1 + \exp\left\{-\left[\gamma_i + \sum_{j=1}^{\infty} \beta_{ij} x_j\right]\right\}} . \qquad (i = 1, 2, 3, \ldots) \tag{II.1}$$

Both of the above systems can be written in the more compact form

$$x_i = \frac{\delta_i}{1 + \exp\left\{-\left[\gamma_i + \sum_{j=1}^{\infty} \beta_{ij} x_j\right]\right\}} , \tag{II.2}$$

where $\delta_i = 1$ for the first of the equations in (II.1), and $\delta_i = \alpha_i^{-1}$ for the second one.

Let X be the Banach space of all infinite dimensional vector functions $x = x(t)$ defined for $t \geqslant 0$ with continuous components $x_1(t)$, $x_2(t)$, $x_3(t), \ldots, x_i(t), \ldots$ such that $0 \leqslant x_i(t) \leqslant 1$ for $t \geqslant 0$ and equipped with the norm

$$\| x \| = \sup_i \max_t | x_i(t) | . \tag{II.3}$$

Consider the operator T defined in X by the equations

$$Tx = z, \qquad z = (z_1, z_2, z_3, \ldots, z_i, \ldots), \qquad x \in X \tag{II.4}$$

where

$$z_i = \frac{\delta_i}{1 + \exp\left\{-\left[\gamma_i + \sum_{j=1}^{\infty} \beta_{ij}\, x_i\right]\right\}}. \tag{II.5}$$

We now assume that

$$\sum_{j=1}^{\infty} |\beta_{ij}| \leqslant 1, \tag{II.6}$$

$$0 < \delta_i \leqslant \delta < 1 + e^{-1-\|\gamma\|} \qquad (i = 1, 2, 3, \ldots).$$

Under these conditions we can easily show that the operator T is a contraction mapping from X into itself such that

$$\| Tx - T\tilde{x} \| < \frac{\delta}{4} \| x - \tilde{x} \| \qquad (x, \tilde{x} \in X). \tag{II.7}$$

Hence, the operator T has a unique fixed point $x^* \in X$, i. e., $Tx^* = x^*$, which can be constructed by the method of successive approximations. Clearly $x^* = (x_1^*, x_2^*, x_3^*, \ldots, x_i^*, \ldots)$ is the unique solution of the infinite system (II.2). Thus, the systems (I.1) and (I.2) admit a unique system of steady-state solutions which can be constructed by iterative methods.

Note that $0 < x_i^* < 1$ for all $i = 1, 2, 3, \ldots$ since $x^* \in X$ by virtue of the above analysis. Further, we have $0 < x_i^* < \frac{1}{\alpha_i}$ for Eqs. (I.2) for $\alpha_i > 1$. Let us also note that conditions (II.6) can be relaxed considerably for the systems (I.1) and (I.2) which contain only finitely many equations.

III. Stability of Solutions

In this section we deal with the stability of steady-state solutions of Eqs. (I.1) and (I.2). First we consider the system (I.1).

Let $x^* = (x_1^*, x_2^*, x_3^*, \ldots, x_i^*, \ldots)$ be the steady-state solution of the system (I.1). Given $\varepsilon > 0$ arbitrarily small, let us choose the initial conditions $x^0 = (x_{1,0}, x_{2,0}, x_{3,0}, \ldots)$, $x_{i,0} = x_i(0)$, in such a way that the inequalities $| x_{i,0} - x_i^* | < \varepsilon$ hold for all $i = 1, 2, 3, \ldots$, i. e.,

$$\| x^0 - x^* \| < \varepsilon. \tag{III.1}$$

Let $x = x(t) \equiv (x_1(t), x_2(t), x_3(t), \ldots, x_i(t), \ldots)$ be the solution of the system (I.1) which satisfies the initial condition $x(0) = x^0$, i. e., Eqs. (I.3). We shall show that

$$\| x(t) - x^* \| < \varepsilon \tag{III.2}$$

for all $t > 0$. For otherwise, there should exist a $t^* > 0$, $t^* = t^*(\varepsilon)$, and an index k, such that

(III.3) $$|x_k(t^*) - x_k^*| = \varepsilon, \quad |x_j(t^*) - x_j^*| \leqslant \varepsilon \quad \text{for} \quad j \neq k.$$

We assume that ε is such that $0 < x_k^* \pm \varepsilon < 1$. We now put

(III.4) $$u(x) = \log \frac{x}{1-x}, \quad u_i = u[x_i], \quad u_{i,0} = u[x_{i,0}], \quad u_i^* = \log \frac{x_i^*}{1-x_i^*}.$$

Then we have the equations

$$\tau \frac{\mathrm{d}u_k}{\mathrm{d}t} + u_k = \gamma_k + \sum_{j=1}^{\infty} \beta_{kj} x_j, \quad u_k^* = \gamma_k + \sum_{j=1}^{\infty} \beta_{kj} x_j^*,$$

from which we find

(III.5) $$\tau \frac{\mathrm{d}v_k}{\mathrm{d}t} + v_k = \sum_{j=1}^{\infty} \beta_{jk} [x_j - x_j^*], \quad v_k(0) = v_{k,0},$$

where

(III.6) $$v_k = u_k - u_k^*, \quad v_{k,0} = u_{k,0} - u_k^*.$$

The solution of Eq. (III.5) at $t = t^*$ is of the form

(III.7) $$v_k(t^*) = v_{k,0}\, \mathrm{e}^{-\frac{t^*}{\tau}} + \frac{1}{\tau} \int_0^{t^*} \mathrm{e}^{-\frac{t^*-\sigma}{\tau}} \left(\sum_{j=1}^{\infty} \beta_{kj} [x_j(\sigma) - x_j^*] \right) \mathrm{d}\sigma.$$

Using the inequalities $\sum_{j=1}^{\infty} |\beta_{kj}| \leqslant 1$ and $|x_j(t) - x_j^*| \leqslant \varepsilon$ for $0 \leqslant t \leqslant t^*$, $j = 1, 2, 3, \ldots$, we can easily verify that

(III.8) $$|v_k(t^*)| < \varepsilon + (|v_{k,0}| - \varepsilon)\, \mathrm{e}^{-\frac{t^*}{\tau}} < |v_{k,0}|.$$

Since $u(x)$ is monotonic, the inequality $|v_k(t^*)| < |v_{k,0}|$ imply that $|x_k(t^*) - x_k^*| < \varepsilon$ which contradicts our assumption (III.3). This contradiction proves the validity of the inequality (III.2) for all $t > 0$.

We now consider Eqs. (I.2). In this case, putting

(III.9) $$w_k = w_k(t) \equiv x_k(t) - x_k^*,$$

we find

(III.10) $$\tau \frac{\mathrm{d}w_k}{\mathrm{d}t} + \alpha_k w_k = f_k(t), \quad w_k(0) = x_k^0 - x_k^*,$$

where

$$\text{(III.11)}\quad f_k(t) = \frac{\exp\left\{-\left[\gamma_k + \sum_{j=1}^{\infty} \beta_{kj} x_j^*\right]\right\} - \exp\left\{-\left[\gamma_k + \sum_{j=1}^{\infty} \beta_{kj} x_j\right]\right\}}{\left(1 + \exp\left\{-\left[\gamma_k + \sum_{j=1}^{\infty} \beta_{kj} x_j\right]\right\}\right)\left(1 + \exp\left\{-\left[\gamma_k + \sum_{j=1}^{\infty} \beta_{kj} x_j^*\right]\right\}\right)}.$$

We can easily verify that

$$\text{(III.12)}\qquad |f_k(t)| < \varepsilon, \qquad (0 \leqslant t \leqslant t^*).$$

On the other hand, the solution of Eq. (III.10) at $t = t^*$ is of the form

$$\text{(III.13)}\qquad w_k(t^*) = w_k(0)\, e^{-\frac{\alpha_k t^*}{\tau}} + \frac{a_k}{\tau}\int_0^{t^*} e^{-\frac{\alpha_k(1-\sigma)}{\tau}} f_k(\sigma)\, d\sigma.$$

By a simple analysis we can show that

$$\text{(III.14)}\qquad |w_k(t^*)| < \varepsilon.$$

But, since $|w_k(t^*)| = |x_k(t^*) - x_k^*| = \varepsilon$, the inequality (III.14) is impossible. Thus, in the case of Eqs. (I.2), the inequality (III.2) holds for all $t > 0$.

The above results prove the stability of the steady-state solutions of Eqs. (I.1) and (I.2) under the conditions (II.6).

Note that if the above systems contain only finitely many equations we can establish more detailed results about the stability of their steady-state solutions. In the next section we deal with this problem.

IV. Stability of Finite Nets

We begin with the finite system

$$\text{(IV.1)}\quad \tau\frac{dx_i}{dt} + \alpha_i x_i = \frac{1}{1 + \exp\left\{-\left[\gamma_i + \sum_{j=1}^{N} \beta_{ij} x_j\right]\right\}}, \quad (i = 1, 2, \ldots, N)$$

where N is a positive integer. Let x_i^*'s be the steady-state solutions of Eqs. (IV.1), that is, the solutions of the equations

$$\text{(IV.2)}\qquad \alpha_i x_i^* = \frac{1}{1 + \exp\left\{-\left[\gamma_i + \sum_{j=1}^{N} \beta_{ij} x_j^*\right]\right\}}, \quad (i = 1, 2, \ldots, N).$$

Clearly $0 < \alpha_i x_i^* < 1$. To study the behavior of the solutions of the system (IV.1), put

$$\text{(IV.3)}\qquad x_i = x_i^* + y_i.$$

Then, by a simple analysis, we find the differential system

$$\frac{dy_i}{dt} = \sum_{j=1}^{N} b_{ij} y_i + Y_i, \quad (i = 1, 2, ..., N) \tag{IV.4}$$

where Y_i's are certain power series in $y_1, y_2, ..., y_N$ containing the second and higher degree terms in y_i's, and

$$b_{ij} = \begin{cases} -\tau^{-1} \alpha_i & \text{if} \quad j = i, \\ \tau^{-1} \beta_{ij} \alpha_i x_i^* (1 - \alpha_i x_i^*) & \text{if} \quad j \neq i. \end{cases} \tag{IV.5}$$

Disregarding the nonlinear terms Y_i's in (IV.4), consider the linear differential system

$$\frac{dy_i}{dt} = \sum_{j=1}^{N} b_{ij} y_j, \quad (i = 1, 2, ..., N) \tag{IV.6}$$

and the corresponding characteristic equation

$$\begin{vmatrix} b_{11}-\lambda & b_{12} & \dots & b_{1N} \\ b_{21} & b_{22}-\lambda & \dots & b_{2N} \\ \dots & \dots & \dots & \dots \\ b_{N1} & b_{N2} & \dots & b_{NN}-\lambda \end{vmatrix} = 0. \tag{IV.7}$$

Expanding Eq. (IV.7), we obtain an equation of the form

$$b_0 \lambda^N + b_1 \lambda^{N-1} + \dots + b_{N-1}\lambda + b_N = 0 . \tag{IV.8}$$

Consider now the determinants

$$\Delta_1 = |b_1|, \quad \Delta_2 = \begin{vmatrix} b_1 & b_0 \\ b_3 & b_2 \end{vmatrix},$$

$$\Delta_3 = \begin{vmatrix} b_1 & b_0 & 0 \\ b_3 & b_2 & b_1 \\ b_5 & b_4 & b_3 \end{vmatrix}, \dots,$$

$$\Delta_N = \begin{vmatrix} b_1 & b_0 & 0 & 0 & \dots & . \\ b_3 & b_2 & b_1 & b_0 & \dots & . \\ b_5 & b_4 & b_3 & b_2 & \dots & . \\ \dots & \dots & \dots & \dots & \dots & . \\ 0 & 0 & 0 & 0 & \dots & b_N \end{vmatrix}. \tag{IV.9}$$

Then, by the well known Routh-Hurwitz criterion, the real parts of the roots λ of Eq. (IV.8) are negative, i. e., the system is stable, provided that all the coefficients $b_0, b_1, \ldots, b_N$ are positive and that all the determinants $\Delta_1, \Delta_2, \ldots, \Delta_N$ are also positive. Otherwise the system is unstable.

Note that, for $N = 1$, the steady-state solution is always stable, since $\lambda = -\frac{\alpha_1}{\tau} < 0$.

For $N = 2$, we have the characteristic equation

$$\lambda^2 - \tau^{-1}(\alpha_1 + \alpha_2)\lambda + \tau^{-2}\alpha_1\alpha_2[1 - \beta_{12}\beta_{21}(1 - \alpha_1 x_1^*)(1 - \alpha_2 x_2^*)] = 0. \tag{IV.10}$$

Therefore we have

$$\Delta_1 = \tau^{-1}(\alpha_1 + \alpha_2) > 0,$$

$$\Delta_2 = \begin{vmatrix} \tau^{-1}(\alpha_1 + \alpha_2) & 1 \\ 0 & \tau^{-2}\alpha_1\alpha_2[1 - \beta_{12}\beta_{21}(1 - \alpha_1 x_1^*)(1 - \alpha_2 x_2^*)] \end{vmatrix} =$$

$$= \tau^{-3}\alpha_1\alpha_2(\alpha_1 + \alpha_2)[1 - \beta_{12}\beta_{21}(1 - \alpha_1 x_1^*)(1 - \alpha_2 x_2^*)].$$

Hence, if

$$1 - \beta_{12}\beta_{21}(1 - \alpha_1 x_1^*)(1 - \alpha_2 x_2^*) > 0 \tag{IV.11}$$

then the system is stable, and, if

$$1 - \beta_{12}\beta_{21}(1 - \alpha_1 x_1^*)(1 - \alpha_2 x_2^*) \leqslant 0 \tag{IV.12}$$

the system is unstable. The stability condition (IV.11) is satisfied whenever β_{12} and β_{21} are in opposite signs, or, if their absolute values are sufficiently small. Note that the steady-state solution is a stable node if (IV.11) holds, is a saddle point if (IV.12) holds. Further, consider the discriminant D_2 of Eq. (IV.10):

$$D_2 = \tau^{-2}[(\alpha_1 - \alpha_2)^2 + 4\alpha_1\alpha_2\beta_{12}\beta_{21}(1 - \alpha_1 x_1^*)(1 - \alpha_2 x_2^*)]. \tag{IV.13}$$

Then we can easily verify that, in the neighborhood of the steady-state solution, the system is non-oscillatory if $D_2 \geqslant 0$, oscillatory if $D_2 < 0$, oscillations being damped when (IV.11) holds, resonant when (IV.12) holds. Note that there is no periodic solutions in the case $N = 2$.

For $N = 3$ we have the characteristic equation

$$b_0\lambda^3 + b_1\lambda^2 + b_2\lambda + b_3 = 0, \tag{IV.14}$$

where

$$b_0 = 1, \quad b_1 = \frac{\alpha_1 + \alpha_2 + \alpha_3}{\tau},$$

(IV.13) $$b_2 = \frac{\alpha_1\alpha_2 + \alpha_2\alpha_3 + \alpha_3\alpha_1}{\tau^2} - b_{12}b_{21} - b_{23}b_{32} - b_{13}b_{31},$$

$$b_3 = \frac{\alpha_1\alpha_2\alpha_3}{\tau^3} - \frac{\alpha_1 b_{23} b_{32} + \alpha_2 b_{13} b_{31} + \alpha_3 b_{12} b_{21}}{\tau}.$$

Hence the system is stable if

(IV.14) $$b_2 > 0, \quad b_3 > 0, \quad b_1 b_2 - b_3 > 0;$$

otherwise it is unstable.

In the case $N = 3$ (and also for $N > 3$), the steady-state solutions may be a node, a saddle, a center or a focus or a saddle-focus according to the specific values of the parameters $\alpha_1, \alpha_2, \alpha_3, b_1, b_2$ and b_3.

We now consider the finite (I.1) system

(IV.15) $$\left(\tau \frac{d}{dt} + 1\right) \log \frac{x_i}{1 - x_i} = \gamma_i + \sum_{j=1}^{N} \beta_{ij} x_j, \quad (i = 1, \ldots, N),$$

where $\beta_{ii} = 0$. Let x_i^* be the steady-state solutions of Eqs. (IV.16). A change of dependent variables according to (IV.2) leads us to Eqs. (IV.4), where

(IV.16) $$b_{ij} = \begin{cases} \tau^{-1} & \text{if} \quad j = i, \\ \tau^{-1} x_i^* (1 - x_i) \beta_{ij} & \text{if} \quad j \neq i. \end{cases}$$

With these new b_{ij}'s we can investigate the stability properties of the steady-state solutions, as above.

Note that for $N = 1$, the steady-state solution is always stable.

For $N = 2$, the characteristic equation is

(IV.17) $$\tau^2\lambda^2 + 2\tau\lambda + [1 - x_1^* x_2^* (1 - x_1^*)(1 - x_2^*) \beta_{12} \beta_{21}] = 0,$$

and

(i) the steady-state solutions are stable if

(IV.18) $$\beta_{12} \beta_{21} x_1^* x_2^* (1 - x_1^*)(1 - x_2^*) < 1,$$

otherwise unstable. Unstable solutions are always non-oscillatory.

(ii) If $\beta_{12} \beta_{21} < 0$ the steady-state solutions are oscillatory (damped);

(iii) There is no periodic solutions in the neighborhood of steady-state solutions.

For $N = 3$, the characteristic equation is

(IV.19) $$\begin{aligned} &\tau^3\lambda^3 + 3\tau^2\lambda^2 + (3 - J_{12} - J_{23} - J_{31})\lambda + \\ &+ [1 - J_{12} - J_{23} - J_{32} - K_1 K_2 K_3 (\beta_{12}\beta_{23}\beta_{31} + \beta_{32}\beta_{21}\beta_{13})] = 0, \end{aligned}$$

where

$$K_i = x_i^* (1 - x_i^*), \quad J_{ij} = K_i K_j \beta_{ij} \beta_{ji} . \tag{IV.20}$$

Accordingly, if

$$J_{12} + J_{23} + J_{32} < \mathrm{Min} \left\{ 3, 1 - K_1 K_2 K_3 \, (\beta_{12}\beta_{23}\beta_{31} + \beta_{32}\beta_{21}\beta_{13}), \right. \\ \left. 4 + \frac{1}{2} K_1 K_2 K_3 \, (\beta_{12}\beta_{23}\beta_{31} + \beta_{32}\beta_{21}\beta_{13}) \right\} \tag{IV.21}$$

the steady-state solutions are stable, otherwise unstable. Particularly, if

$$J_{12} + J_{23} + J_{31} < 0, \quad K_1 K_2 K_3 \, (\beta_{12}\beta_{23}\beta_{31} + \beta_{32}\beta_{21}\beta_{13}) = 0, \tag{IV.22}$$

the steady-state solutions are oscillatory and damped.

Further stability results can be easily obtained by simple algebra.

Note that, for $N = 3$ (also for $N > 3$), the steady-state solutions may be a node, a saddle, a center or a focus or a saddle-focus depending on the specific values of b_i's and K_i's.

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ASUPRA STABILITĂŢII SOLUŢIILOR DE REGIM PERMANENT PENTRU DOUĂ MODELE NEURONALE

(Rezumat)

Se continuă cercetările privind soluţiile ecuaţiilor neuronale stabilite în [15]. După ce se studiază stabilitatea soluţiilor de regim permanent pentru aceste ecuaţii, se analizează mai amănunţit sistemele finite, avînd în vedere importanţa lor pentru cercetările experimentale şi simularea pe calculator.

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К ЗАДАЧЕ ПРОГНОЗИРОВАНИЯ И ИДЕНТИФИКАЦИИ НЕЛИНЕЙНЫХ СИСТЕМ

М. М. КОНСТАНТИНОВ

При решении задач приходится восстановлять характеристики некоторого объекта (идентификация структуры объекта) [1], [2], [3], и его начальные состояния (прогнозирование поведения объекта) на основе косвенных измерений.

В настоящей заметке рассмотрены некоторые аспекты этих проблем для нелинейных динамических объектов, описываемых системами интегро-дифференциальных уравнений, неразрешенных относительно стершей производной.

Пусть движение рассматриваемого объекта описывается системой

$$\dot{u} = f[t\,;\ \alpha,\ u\,;\ \dot{u}\,;\ L(\alpha,\ u)], \tag{1}$$

где $u = (u_1, \ldots, u_q) - q$ - вектор состояния объекта; $\alpha = (\alpha_1, \ldots, \alpha_p)$ — постоянный p - вектор-параметр; $L = (L_1, \ldots, L_\nu)$ — нелинейный интегральный оператор Вольтерра

$$L_j = \int_0^t K_j\,[\alpha,\ u(s)\,;\ s\,;\ t]\,\mathrm{d}s, \qquad (j = 1, \ldots, \nu),$$

$f = (f_1, \ldots, f_q) - q$ - мерная вектор - функция.

Предположим, что в области $t \in I = [0,\ t_0]$; $S \in I$; $\|\alpha\| \leqslant a < \infty$; $\|u\|$, $\|\dot{u}\| \leqslant b < \infty$ где $\|\cdot\|$ некоторая норма соответсвенно в p и q - мерном вещественном пространстве, выполнены условия:

а) $$\mathrm{Det}\left(\delta_q - \frac{\partial f}{\partial \dot{u}}\right) \neq 0, \qquad \delta_q = (\delta_{ij})_1^q.$$

б) Ядра K_j непрерывные и непрерывно дифференцируемые по α, $u(s)$;

в) Функция f непрерывная по всем своим аргументам и непрерывно дифференцируемая по α, u, $\dot{u}$ и L;

г) Задача Коши для системы (1) имеет единственное решение $u = \varphi(\bar{C}, t)$; $\varphi(\bar{C}, 0) = \bar{C}$, $\bar{C} = (\bar{C}_1, ..., \bar{C}_q)$; $\| \varphi \| \leqslant b$, для каждого $\bar{C}$, $\| \bar{C} \| \leqslant b$, на интервале I.

Отметим, что условие г) заведомо выполняется если в рассматриваемой области норма иатрицы $\dfrac{\partial f}{\partial u}$ достаточно малая по сравнении с единицей.

Пусть переменные, доступные измерению (изходные, наблюдаемые переменные), суть величины $z_1, ..., z_m$; $m \leqslant p + q$, связанные с векторами α и u зависимостью

$$z(t) = h[\alpha, u(t)], \quad t \in I \tag{2}$$

где $z = (z_1, ..., z_m)$ а $h = (h_1, ..., h_m)$ – непрерывная и дифференцируемая по α и u функция.

Если положим $x(t) = [\alpha, u(t)]$; $x = (x_1, ..., x_n)$; $n = p + q$, для системы (1) получается

$$\dot{x} = F[t, x, \dot{x}, L(x)], \tag{3}$$

где $F = (F_1, ..., F_n)$:

$$F_i = \begin{cases} 0, & 1 \leqslant i \leqslant p \\ f_{i-p}[t; x; \dot{x}; L(x)], & p + 1 \leqslant i \leqslant n. \end{cases} \tag{4}$$

Пусть в результате соответствующих измерений известна зависимость $z(t)$:

$$z(t) = g(t), \quad g = (g_1, ..., g_m), \quad t \in I, \tag{5}$$

причем степень достоверности измерений i-той компоненты z_i наблюдаемого вектора z характеризуется весовой функцией $p_i(t)$, определенной на I.

При этой постановке проблему прогнозирования и идентификации можно формулировать единным образом в терминах нового обобщенного вектора состояния x:

Требуется найти такое начальное условие

$$C = (C_1, ..., C_n), \quad x(C, 0) = C,$$

решения $x(C, t)$ системы (3), при помощи которого минимизируется функциональ

$$J(C) = \sum_{i=1}^{m} \int_{0}^{t_0} p_i(t)[h_i(x(C, t)) - g_i(t)]^2 \, dt. \tag{6}$$

Предположим, что в согласии с данными измерений и с гипотезой о структуре объекта, искомое начальное условие C мало отличается от вектора C^*, полученного в результате экстраполяции зависимости $h^{-1}[g(t)]$ в окрестности точки $t = 0$ или на основе других соображений:

$$C = C^* + \delta C, \qquad x(t) = x^*(t) + \delta x, \tag{7}$$

где $x^*(t) = x(C^*, t)$ считается известной функцей, а $\delta C = (\delta C_1, ..., \delta C_n)$ и $\delta x = (\delta x_1, ..., \delta x_n)$ являются малыми первого порядка в сравнении с C^* и x^* в смысле соответствующих норм.

Функцию $\delta x(t)$ можно рассматривать как решение некоторого нового интегро - дифференциального уравнения, которого можно получить путем подстановки (7) в (3) и отбрасывания малых второго и высшего порядка:

$$\delta \dot{x} = \left(\delta_n - \frac{\partial F^*}{\partial \dot{x}}\right)^{-1} \frac{\partial F^*}{\partial x} \delta x + \left(\delta_n - \frac{\partial F^*}{\partial \dot{x}}\right)^{-1} \frac{\partial F^*}{\partial L} \delta L, \tag{8}$$

где

$$\frac{\partial F}{\partial x} = \left(\frac{\partial F_i}{\partial x_j}\right)_1^n, \qquad \frac{\partial F}{\partial \dot{x}} = \left(\frac{\partial F_i}{\partial \dot{x}_j}\right)_1^n,$$

$$\frac{\partial F}{\partial L} = \left(\frac{\partial F_i}{\partial L_j}\right)_{n\nu}, \qquad \delta L = (\delta L_1, ..., \delta L_\nu),$$

$$\delta L_j = \sum_{r=1}^{n} \int_{0}^{t} \frac{\partial K_j}{\partial x_r} \delta x_r \, ds, \qquad (j = 1, ..., \nu),$$

а символ * означает, что в соответствующих функциях вместо $x(t)$ следует подставить $x^*(t)$.

Обозначим через $X(t) = (X_{ij}(t))_1^n$ нормированную фундаментальную матрицу системы (8), которую можно получить по меньшей мере путем численного интегрирования. Тогда общее решение системы (8) можно записать в виде

$$\delta x = \delta C X(t). \tag{9}$$

Неизвестные числа δC_i определяются теперь из условий

$$\left.\frac{\partial J}{\partial(\delta C_i)}\right|_{C = C^* + \delta C} = 0, \qquad (i = 1, ..., n). \tag{10}$$

Используя (2), (6) и (9) и после отбрасывания членов высшего порядка малости, (10) принимает вид

$$M\delta C = \mu, \tag{11}$$

где

$$M = (M_{ij})_1^n, \qquad \mu = (\mu_1, \ldots, \mu_n) \tag{12}$$

и

$$M_{ij} = \sum_{l=1}^{m} \sum_{r=\rho=1}^{n,\,n} \int_0^{t_0} p_l \frac{\partial h_l^*}{\partial x_r} \frac{\partial h_l^*}{\partial x_\rho} X_{ri} X_{\rho j}\, dt,$$

$$\mu_i = \sum_{l=1}^{m} \sum_{r=1}^{n} \int p_l (g_l - h_l^*) \frac{\partial h_l^*}{\partial x_r} X_{ri}\, dt, \qquad (i, j = 1, \ldots, n). \tag{13}$$

При условии $\operatorname{Det} M \neq 0$ из системы (11) однозначно определяются поправки δC_i, респективно начальные условия $C_i = C_i^* + \delta C_i$. Предположим теперь, что экспериментальная зависимость (5) имеет вид

$$z(t_s) = g(t_s), \qquad (s = 1, \ldots, N), \tag{14}$$

т. е., что измерения сделаны для некоторой дискретной совокупности точек

$$t_1, \ldots, t_N, \qquad 0 \leqslant t_1 < \ldots < t_N \leqslant t_0.$$

В этом случае система для определения δC тоже имеет вид (11), причем M и μ определяются из

$$M_{ij} = \sum_{l=1}^{m} \sum_{r=\rho=1}^{n,\,n} \sum_{s=1}^{N} p_l(t_s) \frac{\partial h_l^*(t_s)}{\partial x_r} \frac{\partial h_l^*(t_s)}{\partial x_\rho} X_{ri}(t_s) X_{\rho j}(t_s),$$

$$\mu_i = \sum_{l=1}^{m} \sum_{r=1}^{n} \sum_{s=1}^{N} p_l(t_s) [g_l(t_s) - h_l^*(t_s)] \frac{\partial h_l^*(t_s)}{\partial x_r} X_{ri}(t_s), \qquad (i, j = 1, \ldots, n). \tag{15}$$

Рассмотренную процедуру можно использовать для дальнейшего уточнения нулевой гипотезы (вектор C^*) для объекта, если в качестве порождающего решения взять

$$C^{**} = C^* + \delta C, \qquad x^{**}(t) = x^*(t) + \delta x(t). \tag{16}$$

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ASUPRA PROBLEMEI PREVEDERII ŞI IDENTIFICĂRII COMPORTĂRII SISTEMELOR NELINIARE

(Rezumat)

Se consideră cîteva aspecte ale problemei prevederii şi identificării, pentru obiecte dinamice neliniare, descrise de un sistem de ecuaţii integrodiferenţiale, care conţin sub formă implicită derivata de ordinul cel mai ridicat.

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A CONDITION FOR A LINEAR PARTIAL DIFFERENTIAL EQUATION TO HAVE ANALYTIC SOLUTIONS

BY

DORIN CIUMAŞU

In this paper we shall apply the general method for treating such problems, which is studied in [2], to linear partial differential systems, expanding the techniques used by Harris in [3], [4], to linear ordinary differential systems. To every linear partial differential system we shall attach a modified system for which we shall prove the existence of an analytic solution. If certain conditions are satisfied the solutions of modified system will be also solutions of the initial system. As one can find without any difficulties that method is usable and to nonlinear partial differential systems.

The auxiliary notions used here can be found in [1], [5], [6].

1. Let $I \neq \varnothing$, Card $(I) = n$, and $I = N^I$. If $m = (m_i\,;\ i \in I) \in I$, we denote $m = \sum_{i \in I} m_i$ and $m\,! = \prod_{i \in I} m_i!$.

Let also $m' = (m'_i\,;\ i \in I) \in I$. Then $m - m' = (m_i - m'_i\,;\ i \in I) \in I$, where $m - m' = 0$ if $m_i \leqslant m'_i$. We put $m \leqslant m'$ if $m_i \leqslant m'_i$ and $m < m'$ if there exist at least an $i \in I$, so that $m_i < m'_i$.

Let A be a commutative ring, with element 1, of characteristic zero, and $F = A[[X_i\,;\ i \in I]]$ be the ring of formal power series in indeterminates $(X_i\,;\ i \in I)$ over A.

Let $u = (u_m\,;\ m \in I) \in F$ then $\omega(u) = \inf\{|\,m\,|\ ;\ u_m \neq 0\}$, if $u \neq 0$, and $\omega(u) = \infty$ if $u = 0$, is the order of the formal series u.

The strict order of a formal series $u \in F$ is defined by $\Omega(u) = r = (r_i\,;\ i \in I) \in I$, if u, considered as series in the indeterminate X_i, has the order r_i, for each $i \in I$.

We shall say that the formal series $u \in F$ is a quasipolynomial of degree r, if $u_m = 0$ for each $m \geqslant r$. For each formal series $u \in F$ and each $k \in I$, we shall denote by $P_k u$, the quasipolynomial with the property $\Omega(u - P_k u) \geqslant k$.

Hence

$$P_k u(X) = \sum_{m \prec k} u_m X^m,$$

For every $r \in I$ the mapping $D^r : F \to F$, defined by

$$D^r u(X) = \sum_{m \in I} \frac{(m+r)!}{m!} u_{m+r} X^m$$

is the algebraic derivative.

It is obvious that

$$\omega(D^r u) \geqslant \omega(u) - |r|.$$

For every $u \in F$ we set $\| u \| = \exp[-\omega(u)]$. We denote by $T(\omega)$ the topology defined over F by that norm. The topological ring $(F, \| \cdot \|)$ is complete.

Let R_+^* be the set of real strictly positive numbers and $\Gamma = (R_+^*)^I$. If $\alpha = (\alpha_i ; i \in I) \in \Gamma$ we shall denote $|\alpha| = \sum_{i \in I} \alpha_i$.

Let A be a nondiscrete complete valuation ring. We define the mapping

$$\Phi : A[[X_i; i \in I]] \to R[[X_0]], \quad 0 \notin I,$$

such that

$$u(X) = \sum_{m \in I} \frac{u_m}{m!} X^m \in A[[X_i ; i \in I]]$$

has the image

$$\Phi(u(X)) = \sum_{k \in N} \frac{M_k}{k!} X_0^k \in R[[X_0]],$$

where $M_k = \sup\{|u_m| ; |m| = k, m \in I\}$, $k \in N$, $X_0 = \sum_{i \in I} X_i$. (G å r d i n g's formal majorant).

We shall call a convergence bound of the formal series $u(X)$, the convergence radius of the power series $\Phi(u(X))$ and we shall denote it by $\rho(u)$. For any $\alpha \in \Gamma$, $|\alpha| = \beta$ we shall denote $\| u \|_\beta = \Phi u(\alpha)$ and $C_\beta = \{\| u ; \| u \|_\beta < +\infty\}$, $C = \cup C_\beta$, the ring of the convergent series. Then $(C_\beta, \| . \|_\beta)$ is a complete topological ring.

Let B be an open subset of A^I. The mapping $u : B \to A$ is analytic on B, if for any $a \in B$, there exists an element $u_a \in C$ such that $u(x) = u_a(x)$ in some neighbourhood of the point a.

2. We shall mainly consider the case of a linear partial differential equation, a similar method can be used for systems of linear or nonlinear equations.

Let be the equation

$$\sum_{s \in S} a_s(x)\, \mathrm{D}^S\, u(x) = a(x) \ ,$$

where $S \subset I$, $\mathrm{D}^S = \partial^{|s|} / \prod_{i \in I} \partial x_i^{s_i}$, and a_s, $a : A^I \to A$ are analytical function in a some neighbourhood of the origin of A^I.

If $\omega(a_s) \neq 0$ for every $s \in S$ then if we convenient by choose r (for instance such that $| r - q |$ to be maximum if $q = \Omega(a_r) \in I$) the equation can be put in the form

$$x^q \mathrm{D}^r u(x) = \sum_{s \in S} a_s(x)\, \mathrm{D}^s u(x) + a(x), \tag{1}$$

where we have denoted by $a_r(x)$ the function $a_r(x) + x^q$.

The problem can be tackled in a similar way in the case of systems of the form

$$x^{q(j)}\, \mathrm{D}^{r(j)}\, u_j(x) =$$

$$\sum_{k \in S} \sum_{s(j,k) \in S(j,k)} a_{s(j,k)}(x)\, \mathrm{D}^{s(j,k)}\, u_j(x) + a_j(x), \quad j \in J, \tag{2}$$

where J is a finite set of indices, $q(j)$, $r(j) \in I$ and $S(j, k) \subset I$.

In the begining we shall consider this problem in the formal case. In F equation (2) becomes

$$X^q\, \mathrm{D}^r\, u(X) = \sum_{s \in S} a_s\,(X)\, \mathrm{D}^s\, u(X) + a(X), \tag{3}$$

where $a_s(X)$ and $a(X)$ are formal series coresponding to the analytical functions a_s and a, $u(X)$ is an unknown formal series and D^s is the symbol of algebraic derivative in F.

We shall denote

$$Gu(X) = \sum_{s \in S} a_s\,(X)\, \mathrm{D}^s\, u(X), \qquad u(X) = \sum_{m \in I} u_m\, X^m \ . \tag{4}$$

The equation (3) becomes

$$X^q \mathrm{D}^r u(X) = Gu(X) + a(X).$$

Taking into account the definition of algebraic derivative we get

$$\sum_{m \in I} u_{m+r-q}\, X^{m+r-q} = \sum_{m \in I} \frac{(m-q)\,!}{(m+r-q)\,!} (g_m + a_m) X^{m+r-q}. \tag{5}$$

It is obvious that the equations (3) and (5) are equivalent.

In what follows we shall consider the modified equation

$$\sum_{m \geqslant k} u_{m+r-q} X^{m+r-q} = \sum_{m \geqslant k} \frac{(m-q)!}{(m+r-q)!} (g_m + a_m) X^{m+r-q}, \tag{6}$$

where $k \in I$ is fixed.

Define the mapping $L_k : F \to F$, by

$$L_k u(X) = \sum_{m \geqslant k} \frac{(m-q)!}{(m+r-q)!} u_m X^{m+r-q} \tag{7}$$

for every $u \in F$. The the equation (6) becomes

$$u(X) - P_{k+r-q}\, u(X) = L_k G\, u(X) + L_k\, a(X). \tag{8}$$

On the other hand from the equation (6) one gets

$$X^q \mathrm{D}^r [u(X) - P_{k+r-q}\, u(X)] = Gu(X) - P_k Gu(X) + a(X) - P_k a(X),$$

i. e.

$$X^q \mathrm{D}^r u(X) = Gu(X) + a(X) + H_k u(X) \tag{9}$$

where

$$H_k u(X) = X^q \mathrm{D}^r P_{k+r-q}\, u(X) - P_k Gu(X) - P_k a(X). \tag{10}$$

Therefore the equations (6), (8) and (9) are equivalent.

We shall set

$$u(X) = v_k(X) + v(X), \qquad v_k(X) = P_{k+r-q}\, u(X). \tag{11}$$

Hence $\Omega(v) \geqslant k + r - q$.

In this way the equation (8) becomes

$$v(X) = L_k G v(X) + b_k(X), \tag{12}$$

where $b_k(X) = L_k G v_k(X) + L_k a(X)$.

Using that

$$\omega(u\ v) = \omega(u) + \omega(v), \qquad \omega(\sum_i u_i) \geqslant \inf \omega(u_i)$$

for every $u,\ v,\ u_i \in F$, it results that

$$\omega(L_k u) \geqslant |\, k + r - q \,| + \omega(u),$$

therefore the mapping L_k is continuous in $T(\omega)$. Like wise

$$\omega(Gu) \geqslant \inf_{s \in S} \{\, \omega(a_s) - |\, s \,|\, \} + \omega(u).$$

Then

$$\omega(L_k G u) \geqslant | k + r - q | + \inf_{s \in S} \{ \omega(a_s) - | s | \} + \omega(u),$$

therefore the mapping $L_k G$ is a contraction in the topology $T(\omega)$, if

$$| k + r - q | \geqslant | s | - \omega(a_s), \quad s \in S. \tag{13}$$

Hence choosing a $k \in I$ such that the conditions (13) to be satisfied, it follows that for each given formal series $v_k(X)$, there will exist a unique formal series $v(X)$ which fulfils the equation (12) or, what is the same thing the equation (9). That formal series is the limit in $T(\omega)$ of any iterative sequence $\{ v^{(j)} ; j \in N \}$, defined by

$$v^{(j+1)}(X) = L_k G v^{(j)}(X) + L_k G v_k(X) + L_k a(X), \quad j \in N. \tag{14}$$

Suppose now that a_s and a are convergent power series, $\rho(a_s)$, $\rho(a) \geqslant \rho \neq 0$. We remark that from the way in which the mapping G and L_k are defined, it follows that if $\rho(u) \neq 0$, then $\rho(L_k G u) \geqslant \rho(Gu) \geqslant \rho(u)$. Consider a convergent power series $v_k(X)$, with $\rho(v_k) \geqslant \rho$. Let $v^{(0)}$ with $\rho(v^{(0)}) \geqslant \rho$. Then from (14) it follows that $\rho(v^{(1)}) \geqslant \rho$, and by recurrence $\rho(v^{(j)}) \geqslant \rho$, $j \in N$. Therefore if $v^{(0)} \in C_\rho$ then $v^{(j)} \in C_\rho$, $j \in N$. Since this sequence is convergent in $T(\omega)$, and this topology is stictly finer that the topology of the norm $\| \cdot \|_\rho$ in C_ρ, it results that this sequence is also convergent in C_ρ, and its limit is a convergent power series which is just the solution of the equations (12) or (9). Besides that series defines an analytical functions in certain neighbourhood of the origin of A^I, which satisfies in this neighbourhood the equation

$$x^q \mathrm{D}^r u(x) = \sum_{s \in S} a_s(x) \mathrm{D}^s u(x) + a(x) + H_k u(x), \tag{15}$$

where $H_k u(x)$ is defined by (10). Hence

Theorem *Let be the equation* (1) *where* a_s, $a : A^I \to A$, *are analytical functions in some neighbourhood of the origin of* A^I. *Then there exists a analytical* H_k, *quasipolynomial of degree* $k, k \in I$, *which satisfies the condition* (13), *such that the modified equation* (15) *should have an analytical solution in the neighbourhood of origin of* A^I. *The function* H_k *is defined by* (10) *where* $P_{k+r-q}\, u = v_k$, *is a arbitrary analytical function, quasipolynomial of degree* $k + r - q$.

Remarks. 1) If Card $(I) = 1$ then the considered equation is a differential ordinary one and we get the Harris theorem.

2) If the condition $| r - q | \geqslant | s | - \omega(a_s)$, $s \in S$, is fulfilled then we can set $k = 0$ and obtain the case treated in [2].

3) The problem can be approached in a similar way in the case of systems of the form (2). Here the condition (13) is replaced by $| k(j) + r(j) - s(j) | \geqslant | s(i, j) | - \omega(a_{s(i,j)})$, $i, j \in J$, where $k(j) \in I$, for every $j \in J$.

4) A similar method can be used in the case of systems of nonlinear partial differential equations.

5) If $v_k(x)$ is an analytical functions in a neighbourhood of the origin of A^l such that $H_k = 0$, then the solutions of the equation (15) are also solutions of the equation (1). Therefore we have a condition for the equation (1) to have an analytical solution.

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O CONDIŢIE PENTRU CA O ECUAŢIE LINIARĂ CU DERIVATE PARŢIALE SĂ ADMITĂ SOLUŢIE ANALITICĂ

(Rezumat)

Aplicînd metoda generală de tratare a unor astfel de probleme studiată în [2] şi extinzînd tehnica utilizată de Harris în [3], [4] pentru sisteme de ecuaţii ordinare, se obţine o condiţie pentru ca o ecuaţie liniară cu derivate parţiale să admită soluţie analitică. O metodă asemănătoare poate fi utilizată pentru sisteme liniare şi chiar neliniare. Oricărei ecuaţii (1) i se ataşează o ecuaţie corectată (9) pentru care se demonstrează existenţa unei soluţii analitice. Dacă sînt satisfăcute anumite condiţii (observaţia 5) soluţia ecuaţiei corectate este şi soluţie a ecuaţiei date.

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ABOUT THE EQUATION $u_{xyz} + cu = g$

BY

ADRIAN CORDUNEANU

1. Introduction

It is well known that the equation with partial derivatives

$$\frac{\partial^3 u}{\partial x \partial y \partial z}(x, y, z) + c(x, y, z)\, u(x, y, z) = g(x, y, z) \tag{1}$$

where c and g are continuous functions in the domain $-A \leqslant x \leqslant A$, $-B \leqslant y \leqslant B$, $-C \leqslant z \leqslant C$, under the initial conditions

$$u(x, y, 0) = \varphi(x, y), \quad u(0, y, z) = \psi(y, z), \quad u(x, 0, z) = \varkappa(x, z) \tag{2}$$

where φ, ψ, $\varkappa$ belong to the class C^2 and satisfy the conditions

$$\varphi(x, 0) = \varkappa(x, 0), \quad \varphi(0, y) = \psi(y, 0), \quad \psi(0, z) = \varkappa(0, z) \tag{3}$$

is equivalent with the classical Volterra integral equation

$$\begin{aligned} u(x, y, z) = {} & \varphi(x, y) + \psi(y, z) + \varkappa(x, z) - \varphi(x, 0) - \varphi(0, y) - \psi(0, z) + \\ & + \psi(0, 0) + \iiint_{0\,0\,0}^{x\,y\,z} g(\xi, \eta, \zeta)\, d\xi\, d\eta\, d\zeta - \iiint_{0\,0\,0}^{x\,y\,z} c(\xi, \eta, \zeta)\, u(\xi, \eta, \zeta)\, d\xi\, d\eta\, d\zeta. \end{aligned} \tag{4}$$

In this Note we show that the relatively more complicated problem when the values of u and their derivatives on certain surface Σ are given, may be reduced to a problem if the type (1) + (2).

2. Definitions and Notations. Theorem

We denote by Σ the surface of the equation $z = \alpha(x, y)$, $-A \leqslant x \leqslant A$, $-B \leqslant y \leqslant B$, where the function α is of the class C^2 and $-C < \alpha < C$.

We suppose that Σ has the property that every parallel to the coordinate axis has in commonly with him at most one point.

We denote by Δ the set of all points $M(x, y, z)$ having the property that the parallels with Ox, Oy, Oz passing through M rejoin Σ and we put $D = \text{int}\,\Delta$. We also denote by $\Omega(M)$ the volume limited by Σ and the parallel planes with xOy, xOz, yOz, passing through M and by $\Sigma\,(M)$ the part of the surface Σ determined by the same planes.

We remember that $C(D)$ is the class of all continuous functions in D and $C^k(D)$ is the class of all functions possesing continuous partial derivatives of the order k in D.

Finally, we denote by $\mathcal{U}(D)$ the class of all functions $u = u(x, y, z)$ having continuous partial derivatives of the second order in D and such that $(u_{xy})_z = (u_{xz})_y = (u_{yz})_x = u_{xyz}$ is continuous, i. e.

$\mathcal{U}(D) = \{u\,;\ u \in C^2(D)$ and u_{xyz} is uniquely determined and continuous in $D\}$.

Theorem. *Under the conditions cited above on the coefficients c and g and the surface* Σ, *the equation* (1) *with the conditions*

$$u(x, y, \alpha) = \varphi(x, y),\ u_z\,(x, y, \alpha) = \varphi_1(x, y), u_{xz}\,(x, y, \alpha) = \varphi_2(x, y) \tag{5}$$
$$u_{yz}\,(x, y, \alpha) = \varphi_3(x, y), u_{zz}\,(x, y, \alpha) = \varphi_4(x, y)$$

where $\varphi \in C^2((-A,\ A)\times(-B,\ B))$ *and* $\varphi_i \in C((-A,\ A)\times(-B,\ B))$ *for* $i = 1,\ 2,\ 3,\ 4$ *has a unique solution* $u \in \mathcal{U}(D)$.

Proof. Firstly, we remark that is not necessary to give the values of all derivatives of u on the surface Σ, because by

$$\left\{\begin{aligned} & u_x\,(x, y, \alpha) + u_z\,(x, y, \alpha)\,\frac{\partial\alpha}{\partial x} = \frac{\partial\varphi}{\partial x}\,, \\ & u_y\,(x, y, \alpha) + u_z\,(x, y, \alpha)\,\frac{\partial\alpha}{\partial y} = \frac{\partial\varphi}{\varphi y}\,, \end{aligned}\right. \tag{6}$$

it follows that u_x and u_y are known on Σ ; similarly, from the relations

$$
(7)\quad \begin{cases}
u_{xx}(x,y,\alpha) + 2u_{xz}(x,y,\alpha)\dfrac{\partial\alpha}{\partial x} + u_{zz}(x,y,\alpha)\left(\dfrac{\partial\alpha}{\partial x}\right)^2 + u_z(x,y,\alpha)\dfrac{\partial^2\alpha}{\partial x^2} = \dfrac{\partial^2\varphi}{\partial x^2}, \\[2ex]
u_{xy}(x,y,\alpha) + u_{xz}(x,y,\alpha)\dfrac{\partial\alpha}{\partial y} + u_{yz}(x,y,\alpha)\dfrac{\partial\alpha}{\partial x} + u_{zz}(x,y,\alpha)\dfrac{\partial\alpha}{\partial x}\dfrac{\partial\alpha}{\partial y} + \\[2ex]
\qquad\qquad + u_z(x,y,\alpha)\dfrac{\partial^2\alpha}{\partial x\partial y} = \dfrac{\partial^2\varphi}{\partial x\partial y}, \\[2ex]
u_{yy}(x,y,\alpha) + 2u_{yz}(x,y,\alpha)\dfrac{\partial\alpha}{\partial y} + u_{zz}(x,y,\alpha)\left(\dfrac{\partial\alpha}{\partial y}\right)^2 + \\[2ex]
\qquad\qquad + u_z(x,y,\alpha)\dfrac{\partial^2\alpha}{\partial y^2} = \dfrac{\partial^2\varphi}{\partial y^2}
\end{cases}
$$

it follows that u_{xx}, u_{xy}, u_{yy} are also known on Σ.

If we put

$$
(8)\qquad \mathcal{L}u = \frac{\partial^3 u}{\partial x\partial y\partial z} + cu \quad \text{and} \quad \mathcal{M}v = -\frac{\partial^3 v}{\partial x\partial y\partial z} + cv,
$$

or $u, v \in \mathcal{U}(D)$, we get

$$
(9)\qquad \begin{aligned}
&\iiint_{\Omega} (v\mathcal{L}u - u\,\mathcal{M}v)\,d\omega = \\
&= \frac{1}{3}\iiint_{\partial\Omega} [(uv)_{yz}\cos\alpha + (uv)_{xz}\cos\beta + (uv)_{xy}\cos\gamma]\,d\sigma,
\end{aligned}
$$

for every domain $\Omega \subset D$, $\bar{n} = \cos\alpha\bar{i} + \cos\beta\bar{j} + \cos\gamma\bar{k}$ being the outward normal to $\partial\Omega$ = the boundary of Ω (Fig. 1).

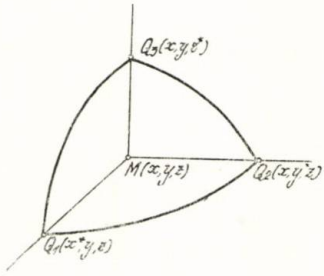

Fig. 1

If $\Omega = \Omega(M)$ and u is a solution of (1) + (5), we obtain

$$\iiint\limits_{\Omega(M)} vg\,\mathrm{d}\omega = \iiint\limits_{\Omega(M)} u\mathcal{M}\,v\mathrm{d}\omega + \frac{1}{3}\iint\limits_{\Sigma(M)} [(uv)_{yz}\cos\alpha + (uv)_{xz}\cos\beta +$$

$$+ (uv_{xy}\cos\gamma]\,\mathrm{d}\sigma + \tag{10}$$

$$+ \sum_{i \neq j} \frac{1}{3} \iiint\limits_{MQ_iQ_j} [(uv)_{yz}\cos\alpha + (uv)_{xz}\cos\beta + (uv)_{xy}\cos\gamma]\,\mathrm{d}\sigma,$$

or, if we apply the Stokes formula

$$\iiint\limits_{\Omega(M)} vg\mathrm{d}\omega = \iiint\limits_{\Omega(M)} u\mathcal{M}v\,\mathrm{d}\omega + \frac{1}{3}\iint\limits_{\Sigma(M)} [(uv)_{yz}\cos\alpha + (uv)_{xz}\cos\beta +$$

$$+ (uv)_{xy}\cos\gamma]\,\mathrm{d}\sigma - (uv)(M) + \frac{1}{3}\Big[uv(Q_1) + (uv)(Q_2) + \tag{11}$$

$$+ (uv)(Q_3) + \int\limits_{\overset{\frown}{Q_1Q_3}} (uv)_x\,\mathrm{d}x + \int\limits_{\overset{\frown}{Q_2Q_1}} (uv)_y\,\mathrm{d}y + \int\limits_{\overset{\frown}{Q_3Q_2}} (uv)_z\,\mathrm{d}z \Big]$$

which finally imply that

$$(uv)(M) = \frac{1}{3}\Big[(uv)(Q_1) + (uv)(Q_2) + (uv)(Q_3) + \int\limits_{\overset{\frown}{Q_1Q_3}} (uv)\,\mathrm{d}x +$$

$$+ \int\limits_{\overset{\frown}{Q_2Q_1}} (uv)_y\,\mathrm{d}y + \int\limits_{\overset{\frown}{Q_3Q_2}} (uv)_z\,\mathrm{d}z \Big] + \frac{1}{3}\iint\limits_{\Sigma(M)} [(uv)_{yz}\cos\alpha + \tag{12}$$

$$+ (uv)_{xz}\cos\beta + (uv)_{xy}\cos\gamma]\,\mathrm{d}\sigma - \iiint\limits_{\Omega(M)} vg\,\mathrm{d}\omega + \iiint\limits_{\Omega(M)} u\mathcal{M}\,v\mathrm{d}\omega,$$

for every $v \in \mathcal{U}(D)$.

As and in the Riemann method for the hyperbolic equation of the second order ([1], pp. 217—228), we choose the function v such that $\mathcal{M}v = 0$ in $\Omega(M)$ and $v = 1$ on the planes MQ_1Q_2, MQ_2Q_3 and MQ_3Q_1. But in this case it is clear that $v = v(\xi, \eta, \zeta, ; x, y, z)$, where ξ, η, ζ are current coordinates in the space and x, y, z are the coordinates of the fixed point $M \in D$. Thus, the conditions for v are

$$
(13)\qquad \begin{cases} v_{\xi\eta\zeta}(\xi,\ \eta,\ \zeta\,;\ x,\ y,\ z) = c(\xi,\ \eta,\ \zeta)\, v(\xi,\ \eta,\ \zeta\,;\ x,\ y,\ z), \\ v(x,\ \eta,\ \zeta\,;\ x,\ y,\ z) = 1, \\ v(\xi,\ y,\ \zeta\,;\ x,\ y,\ z) = 1, \\ v(\xi,\ \eta,\ z\,;\ x,\ y,\ z) = 1 \end{cases}
$$

and the formula (12) becomes

$$
u(x, y, z) = \frac{1}{3}\Big[\ \varphi(Q_1) + \varphi(Q_2) + \varphi(Q_3) + \int\limits_{\overset{\frown}{Q_1Q_3}} (uv)_\xi\, \mathrm{d}\xi +
$$

$$
(14)\quad + \int\limits_{\overset{\frown}{Q_2Q_1}} (uv)_\eta\, \mathrm{d}\eta + \int\limits_{\overset{\frown}{Q_3Q_2}} (uv)_\zeta\, \mathrm{d}\zeta\Big] + \frac{1}{3} \iint\limits_{\Sigma(M)} [(uv)_{\eta\zeta} \cos\alpha + (uv)_{\xi\zeta} \cos\beta +
$$

$$
+ (uv)_{\xi\eta} \cos\gamma]\, \mathrm{d}\sigma - \iiint\limits_{\Omega(M)} v(\xi, \eta, \zeta\,;\ x, y, z)\, g(\xi, \eta, \zeta)\, \mathrm{d}\omega.
$$

From (13) we deduce that v is uniquely determined as the solution of the following equation

$$
(15)\qquad v(\xi, \eta, \zeta, x, y, z) = 1 + \iiint\limits_{x\,y\,z}^{\xi\,\eta\,\zeta} c(\rho, \sigma, \tau)\, v(\rho, \sigma, \tau\,;\ x, y, z)\, \mathrm{d}\rho \mathrm{d}\sigma \mathrm{d}\tau,
$$

where $M = M(x,\ y,\ z) \in D$ and $P = P(\xi,\ \eta,\ \zeta) \in \Omega(M)$. The function v may be obtained by the method of the succesive approximations and clearly belongs to the class $\mathcal{U}(\Omega(M))$ considered as function in ξ, η, ζ. If we take into account a result of [2] concerning the Volterra equation in several variables, we conclude that v belongs to the class $\mathcal{U}(D)$ considered as function in x, y, z.

Thus from (14) we deduce that if the problem (1) + (5) has solution in the class $\mathcal{U}(D)$, this solution is unique.

To show that (14) indeed is the required solution for (1) + (5), we remark that after a suitable change of the unknown function, we can reduce the problem (1) + (5) to

$$
(16)\qquad \begin{cases} \dfrac{\partial^3 u}{\partial x \partial y \partial z}(x, y, z) + c(x, y, z)\, u(x, y, z) = g(x, y, z), \\ u(x, y, \alpha) = u_z(x, y, \alpha) = u_{xz}(x, y, \alpha) = u_{yz}(x, y, \alpha) = u_{zz}(x, y, \alpha) = 0 \end{cases}
$$

and then from (14) it follows

$$u(x, y, z) = -\iiint\limits_{\Omega(M)} v(\xi, \eta, \zeta\,;\; x, y, z)\, g(\xi, \eta, \zeta)\, \mathrm{d}\omega. \tag{17}$$

We have

$$\left\{\begin{aligned} \frac{\partial u}{\partial x}(x, y, z) &= -\iiint\limits_{\Omega(M)} \frac{\partial v}{\partial x}(\xi, \eta, \zeta\,;\; x, y, z)\, g(\xi, \eta, \zeta)\, \mathrm{d}\omega + \\ &\quad + \iint\limits_{\Omega(M\,;\,x)} v(x, \eta, \zeta\,;\; x, y, z)\, g(x, \eta, \zeta)\, \mathrm{d}\eta\, \mathrm{d}\zeta = \\ &= -\iiint\limits_{\Omega(M)} \frac{\partial v}{\partial x}(\xi, \eta, \zeta\,;\; x, y, z)\, g(\xi, \eta, \zeta)\, \mathrm{d}\omega + \iiint\limits_{\Omega(M\,;\,x)} g(x, \eta, \zeta)\, \mathrm{d}\eta\, \mathrm{d}\zeta, \end{aligned}\right. \tag{18}$$

where by $\Omega(M\,;\,x)$ we have denoted the projection on yOz of MQ_2Q_3.

Similarly we obtain

$$\begin{aligned} \frac{\partial^2 u}{\partial x \partial y}(x, y, z) &= -\iiint\limits_{\Omega(M)} \frac{\partial^2 v}{\partial x \partial y}(\xi, \eta, \zeta\,;\; x, y, z)\, g(\xi, \eta, \zeta)\, \mathrm{d}\omega - \\ &\quad - \int_{z}^{z^*} g(x, y, \zeta)\, \mathrm{d}\zeta \end{aligned} \tag{19}$$

and finally

$$\frac{\partial^3 u}{\partial x \partial y \partial z}(x, y, z) = -\iiint\limits_{\Omega(M)} \frac{\partial^3 v}{\partial x \partial y \partial z}(\xi, \eta, \zeta\,;\; x, y, z)\, g(\xi, \eta, \zeta)\, \mathrm{d}\omega + g(x, y, z). \tag{20}$$

The condition that $u = u(x, y, z)$ satisfies the equation (1) in the point $M(x, y, z) \in D$, is reduced to

$$\frac{\partial^3 v}{\partial x \partial y \partial z}(\xi, \eta, \zeta\,; x, y, z) + c(x, y, z)\, v(\xi, \eta, \zeta\,; x, y, z) = 0 \tag{21}$$

for $(\xi, \eta, \zeta) \in \Omega(M)$.

If we take into account (15), we obtain that

$$\frac{\partial^3 v}{\partial x \partial y \partial z}(\xi, \eta, \zeta; x, y, z) + c(x, y, z)\, v(\xi, \eta, \zeta; x, y, z) =$$

$$(22) \qquad = \iiint\limits_{\Omega(M)} c(\rho, \sigma, \tau)\left(\frac{\partial^3 v}{\partial x \partial y \partial z}(\rho, \sigma, \tau; x, y, z) + \right.$$

$$\left. + c(x, y, z)\, v(\rho, \sigma, \tau; x, y, z)\right) d\rho\, d\sigma\, d\tau$$

for $(\xi, \eta, \zeta) \in \Omega(M)$, i.e. the function $\psi = \psi(\xi, \eta, \zeta) = v_{xyz}(\xi, \eta, \zeta; x, y, z) + c(x, y, z)\, v(\xi, \eta, \zeta; x, y, z)$ satisfies a Volterra homogeneous linear equation, thus $\psi(\xi, \eta, \zeta) = 0$ in $\Omega(M)$ and the function $u = u(x, y, z)$ satisfies (1) in D, if we remember that $M(x, y, z)$ was an arbitrary point in D.

The conditions at the boundary (16) are also satisfied by the function u given by (17) and so the theorem is proved.

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DESPRE ECUAŢIA $u_{xyz} + cu = g$

(Rezumat)

Coeficienţii c şi g fiind presupuşi continui în domeniul $-A \leqslant x \leqslant A$, $-B \leqslant y \leqslant B$, $-C \leqslant z \leqslant C$, se rezolvă ecuaţia de mai sus în ipoteza că sînt date valorile funcţiei u şi ale derivatelor sale pe o suprafaţă Σ ce satisface anumite proprietăţi (şi este situată în domeniul considerat). Metoda de rezolvare se aseamănă cu aşa numita metodă a funcţiei lui Riemann pentru ecuaţia hiperbolică cu derivate parţiale de ordinul doi.

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SYSTÈMES DIFFÉRENTIELS POSSÉDANT LA STRUCTURE COMPLEXE

PROBLÈMES À LA FRONTIÈRE POUR CERTAINES CLASSES D'ÉQUATIONS INTÉGRO-DIFFÉRENTIELLES NON LINÉAIRES DE MANGERON

PAR

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1. Les auteurs, tout en tenant compte de l'importance des équations de Mangeron, utilisées, par ex., dans les desseins automatiques des surfaces de forme quelconque [1], et de l'intérêt de toute une série de travaux consacrés à ces équations, exposent bien succinctement dans ce qui suit quelques-uns des points saillants de leurs recherches récentes concernant les systèmes intégro-différentiels non linéaires de Mangeron [2]...[4].

2. Considérons le système constitué du problème généralisé de Goursat

$$(1)\qquad \begin{cases} u(x,\alpha)=\varphi\left[x,\displaystyle\int_{\alpha}^{\gamma}P(x,\tau,u(x,\tau))\,\mathrm{d}\tau\right], & \varphi[\cdot]_{x=a}=p=\text{const},\\[2ex] u(a,t)=\psi\left[t,\displaystyle\int_{a}^{b}Q(t,\xi,u(\xi,t))\,\mathrm{d}\xi\right], & \psi[\cdot]|_{t=\alpha}=p=\text{const},\end{cases}$$

concernant la classe suivante d'équations de Mangeron

$$(2)\qquad u''_{xt}(x,t)=f\left[x,t,u(x,t),\int_{a}^{x}M(x,t,\xi,u)\mathrm{d}\xi,\int_{\alpha}^{t}N(x,t,\tau,u)\mathrm{d}\tau\right],$$

où $\varphi(x, v_1)$, $P(x, \tau, v_2)$, $\psi(t, v_3)$, $Q(t, \xi, v_4)$, $f(x, t, v_5, v_6, v_7)$, $M(x, t, \xi, v_4)$, $N(x, t, \tau, v_2)$ sont des fonctions définies et continues dans le domaine $D = \{a \leqslant x, \xi \leqslant b, \alpha \leqslant t, \tau \leqslant \gamma, 0 \leqslant | v_i \leqslant r_i, i = \overline{1, 7}, r = \text{const}\}$ et y satisfont par rapport aux variables v_i les conditions de Lipschitz, dont les coefficients correspondants supposés continus dans D sont $L_\varphi(x)$, $L_P(x, \tau)$, $L_\psi(t)$, $L_Q(t, \xi)$, $L_{if}(x, t)$, $i = \overline{1, 3}$, $L_M(x, t, \xi)$ et $L_N(x, t, \tau)$.

En utilisant la méthode de transformation des systèmes intégro-différentiels dans les équations intégrales équivalentse [5]...[6], on aboutit à transformer le système (1), (2) dans l'équation intégrale non linéaire équivalente, que l'on peut écrire brièvement sous forme

$$u(x, t) = A[u]. \tag{3}$$

Après avoir choisi pour norme la définition

$$\| \cdot \| = \max | \cdot |$$

et avoir pris deux fonctions arbitraires u_1, $u_2 \in D$, on en déduit de (3) l'inégalité

$$\| A[u_2] - A[u_1] \| \leqslant \nu \| u_2 - u_1 \|. \tag{4}$$

D'où le suivant

Théorème 1. *Si, dans* (4), *on a* $\nu < 1$, *l'opérateur* $A[\ .\ ]$ *applique l'ensemble des fonctions continues du domanie* D *dans l'une de ses parties et par conséquent l'équation* (3) *et donc le problème* (1), (2) *possèdent, d'après le principe du point fixe de Banach, une solution continue unique, ayant les dérivées* $u'_x(x, t)$, $u'_t(x, t)$, $u''_{xt}(x, t)$, $\forall(x, t) \in D$, *elles aussi continues dans* D. *La construction effective de la solution de ce problème se réalise grâce a la convergence du processus de recurrence*

$$u_n(x, t) = A[u_{n-1}], \quad (n = 1, 2, \ldots), \quad \lim_{n \to \infty} u_n(x, t) = u(x, t). \tag{5}$$

L'évaluation de l'erreur commise $\|u_n - u\|$ est donnée par exemple poar le suivant

Théorème 2. *Dans les conditions ci-dessus a lieu l'inégalité*

$$\|u_n - u\| \leqslant \|r_n(x, t)\| : (1 - \nu) \equiv \rho_n, \tag{6}$$

où l'on a mis

$$r_n(x, t) \equiv u_n(x, t) - A[u_n]. \tag{7}$$

3. Considérons maintenant la classe suivante d'équations intégro-différentielles non linéaires de Mangeron [7]

$$M[u(x, t)] = f\Big[x, t, u(x, t), \int_a^x K(x, t, \xi, u(\xi, t, (u'_t(\xi, t))\, d\xi, \int_a^t P(x, t, \tau, u(x, \tau), u'_x(x, \tau))\, d\tau\Big], \tag{8}$$

où $M[u(x, t)] \equiv \dfrac{\partial^2 u(x, t)}{\partial x\, \partial t}$ est la dérivée totale du premier ordre au sens de M. M. Picone [8], la fonction inconnue $u(x, t)$ étant soumise aux conditions intégrales-fonctionnelles à la frontière

$$(9) \qquad u(x, \alpha) = \varphi(x), \quad u(a, t) = \psi\left[t, \int_\alpha^t \int_a^b N(\xi, \tau, M[u(\xi, \tau)])\, d\xi\, d\tau\right],$$

$$\varphi(a) = \psi(\alpha, 0) = P.$$

On suppose que les fonctions $f(x, t, s_1, s_2, s_3)$, $K(x, t, \xi, s_4, s_5)$, $\psi(t, s_8)$, $P(x, t, \tau, s_6, s_7)$ et $N(\xi, \tau, s_9)$ sont définies, continues et lipschitziennes par rapport aux variables $s_i (i = \overline{1,9})$, respectivement, dans le domaine $D = \{a \leqslant x, \ \xi \leqslant b, \ \alpha \leqslant t, \ \tau \leqslant \gamma, \ 0 \leqslant |s_i| \leqslant r_i = \text{const}, \ (i = \overline{1, 9})$ et soient $L_{if}(x, t)$ $(i = 1, 2, 3)$, $L_{iK}(x, t, \xi)$ $(i = 1, 2)$, $L_{iP}(x, t, \tau)$ $(i = 1, 2)$, $L_N(\xi, \tau)$ et $L_\psi(t)$ les coefficients correspondants, supposés continus dans D de Lipschitz. On suppose en outre que les fonctions $\varphi(x)$ et $\psi(t, s_8)$ sont continuement différentiables dans D et possèdent les dérivées $\psi'_t(t, s_8)$ et $\psi'_{s_8}(t, s_8)$ continues et lipschitziennes dans le même domaine, ayant pour coefficients de Lipschitz $L_{\psi'_t}(t)$ et $L_{\psi'_{s_8}}(t)$. À la suite de l'utilisation de la transformation intégrale

$$(10) \qquad \begin{aligned} u(x, t) &= h(x) + \psi\left[t, \int_a^t \int_\alpha^b N(\xi, \tau, v(\xi, \tau))\, d\xi\, d\tau\right] + \\ &+ \int_a^x \int_\alpha^t v(\xi, \tau)\, d\tau\, d\xi \equiv H[x, t, v], \quad h(x) = \varphi(x) - p, \end{aligned}$$

le problème (1), (2) se réduit à l'intégration de l'équation intégrale non linéaire équivalente que l'on peut écrire, en renonçant ici aux développements algorithmiques, d'une manière assez concise, sous forme

$$(11) \qquad v(x, t) = A_1[x, t, v].$$

À la suite du fait que l'on a, pour $\bar{v}_1$ et $\bar{v}_2 \in D$,

$$(12) \qquad \| A_1[x, t, \bar{v}_2] - A_1[x, t, \bar{v}_1] \| \leqslant \mu \| \bar{v}_2 - \bar{v}_1, \|$$

on en déduit le suivant

Théorème 3. *L'équation intégrale* (11) *possède dans le domaine* D, *pour* $\mu < 1$, *une solution continue unique* $v(t, x)$ *et donc le système intégro-différentiel non linéaire de Mangeron* (8), (9) *admet lui aussi une solution unique* $u(x, t)$, *continue avec ses dérivées* $u'_x(x, t)$, $u'_t(x, t)$ *et* $u''_{xt}(x, t)$ *dans* D.

Les solutions respectives approchées $v_n(x, t)$ et $u_n(x, t)$ s'obtiennent, à des erreurs fixées d'avance près, grâce aux formules de recurrence

$$v_n(x, t) = A_1[x, t, v_{n-1}], \qquad n = 1, 2, \ldots, \tag{13}$$

$$u_n(x, t) = H[x, t, v_n], \qquad n = 0, 1, \ldots \tag{14}$$

et l'on a $\lim\limits_{n\to\infty} u(x, t) = u(x, t)$.

En ce que regarde le problème de l'évaluation des erreurs commises a lieu le suivant

Théorème 4. *Étant donné un nombre positif ε, l'expression de l'erreur commise en choisisans pour la solution $u(x, t)$ une solution approcheé $u_n(x, t)$est donnée, à ε près par*

$$\|u - u_n\| \leqslant$$

$$\left\| L_\psi(t) \int_\alpha^t \int_a^b L_N(\xi, \tau)\, d\xi\, d\tau + (x - a)(t - \alpha) \right\| \mu^n \| r_0(x, t) \| : (1 - \mu) \equiv \rho\mu^n, \tag{15}$$

$$r_{n-1}(x, t) \equiv v_{n-1}(x, t) - A_1[x, t, v_{n-1}], \quad n = 1, 2, \ldots,$$

ou l'on a choisi $n \leqslant \ln(\varepsilon : \rho) : \ln \mu$.

Dans quelques-unes de nos Notes prochaines nous exposérons d'autres extensions intéressantes concernant différentes classes d'équations polyvibrantes de Mangeron, examinerons les cas de linéarisation correspondantes et indiquerons quelques autres domaines d'application de ces équations.

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SISTEME MATEMATICE CU STRUCTURĂ COMPLEXĂ

Probleme de contur pentru unele clase de ecuaţii integro-diferenţiale neliniare tip Mangeron

(Rezumat)

Autodrii discută capacitatea de determinare a soluţiilor unor sisteme integro-diferenţiale de tip Mangeron şi stabilesc o serie de teoreme de existenţă, unicitate şi stabitate a soluţiilor unor atare sisteme. Se dau formule de evaluare a erorilor comise în cazul soluţiilor aproximative determinate prin recurenţă.

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A FUNCTIONAL EQUATION ASSOCIATED WITH A NON-LINEAR RECURSION

BY

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0. Introduction

This note continues the investigation of a problem initiated in a previous paper [1]. We are interested in the nature of sequences satisfying the recursion relation

$$x_{n-1}\, x_{n+1} = x_n - \alpha, \tag{0.1}$$

where α is an arbitrary real number. We have already shown that, for each $\alpha \in (0,\ 1/4]$, there exists a sequences $\{x_n,\ n = 0, 1, 2, \ldots\}$ of positive numbers satisfying (0.1) with the initial condition $x_0 = \alpha$. We now wish to prove that there is only one such sequence for each α; we also wish to investigate the nature of doubly infinite sequences $\{x_n,\ n = 0,\ \pm 1,\ \pm 2, \ldots\}$ satisfying (0.1).

We note that each solution of (0.1) is completely determined, as for it is determinate, by any pair of consecutive elements x_n and x_{n+1}. Further, moving along the sequence from x_{n-1}, x_n to x_{n+1} is equivalent to moving from a point $(x,\ y)$ to the point $(y,\ z)$ through the transformation $xz = y - \alpha$.

We study invariants associated with, and orbits generated by, a class of transfosmation sof which the one we are interested in is a member.

1. A functional Eequation Associated with a Class of Recursion Relations

Note that (0.1) is a special case of the relation $x_{n+1} = g(x_{n-1},\ x_n)$, where g is some function on R_2. The problem of finding invariants associated with the transformation $\{(x,\ y) \to (y,\ z),\ z = g(x,\ y)\}$ is

Problem A. Given a function g on R_2, we wish to find a function f on R_2 such that $f(x, y) = f[y, g(x, y)]$.

Alternatively, the problem of finding orbits associated with the transformation is

Problem B. Given a function g on R_2, we wish to find a function F on R_2 such that $F(x, y) = 0$ and $z = g(x, y)$ imply $F(y, z) = 0$.

We shall restrict ourselves to a very special case of B and a special form of g, namely, $g(x, y) = g(y)/x$; this is adequate for dealing with the recursion (0.1). We now prove a theorem which deals with a functional equation, the solution of which is a one-parameter family of functions on R_2 which define the orbits associated with the transformation.

Remark 1.1. Note that if $x = 0$, either there is no z satisfying $xz = g(y)$ [in case $g(y) \neq 0$] or every z satisfies the equation [in case $g(y) = 0$]; this reflects the fact that the origin plays a special role in the recursion $x_{n-1} x_{n+1} = g(x_n)$.

Theorem. *Let a, b, g be functions on R_1 and F a function on R_2 such that*

$$F(x, y)\, a(z) - a(x)\, F(y, z) = (x - z) b(y)\, [xz - g(v)] = \tag{1.1}$$
$$= (x - z)[xz\, b(y) - c(y)], \quad x, y, z \in R_1,$$

here $c(y) = b(y)\, g(y)$; it is assumed that a and g are not identically constant. Then a, b, c and F are polynomials, with

$$a(x) = a_0 x, \tag{1.2}$$

$$b(x) = b_0 + b_1 x + b_2 x^2, \tag{1.3}$$

$$c(x) = b(x)\, g(x) = c_0 + c_1 x + b_0 x^2, \tag{1.4}$$

$$F(x, y) = F_\theta(x, y) = \frac{1}{a_0} [c_0 + c_1(x + y) + \theta\, xy + b_0(x^2 + y^2) + \tag{1.5}$$
$$+ b_1 xy(x + y) + b_2 x^2 y^2],$$

where θ is arbitrary.

Proof. By assumption, there is an x_0 such that $a(x_0) \neq 0$. Taking $x = z = x_0$ in (1.1), we get $F(x_0, y) = F(y, x_0)$ for all y; by interchanging x and z and comparing the new equation with (1.1), we obtain

$$a(x)\, \{F(z, y) - F(y, z)\} = a(z)\, \{F(y, x) - F(x, y)\},$$

from which, on setting $z = x_0$, we have

$$F(x, y) = F(y, x) \quad \text{for all} \quad (x, y). \tag{1.6}$$

On interchanging x and y in (1.1) and using the symmetry of F, we get

$$a(z)\, F(x, y) - a(y)\, F(x, z) = (y - z)[yz\, b(x) - c(x)], \tag{1.7}$$

and combining this with (1.1),

$$a(x)\,F(y,\ z) - a(y)\,F(x,\ z) = (y - z)\,[yz\,b(x) - c(x)] - \\ - (x - z)\,[xzb(y) - c(y)]. \tag{1.8}$$

If we make the permutation $x \to y \to z \to x$ in (1.1), the equation so obtained, together with (1.8), gives us

$$(x - y)\,xy\,b(z) + (y - z)\,yz\,b(x) + (z - x)\,zxb(y) = \\ = (x - y)c(z) + (y - z)c(x) + (z - x)\,c(y), \tag{1.9}$$

which, for $z = 0$, becomes

$$y[c(x) - c(0) - b(0)x^2] = x[c(y) - c(0) - b(0)y^2]. \tag{1.10}$$

This identity implies (1.4), which on substitution in (1.9) yields (1.3).

Finally, setting $x = 0$ in (1.1), we obtain

$$a(z)\,F(0,\ y) - a(0)\,F(y,\ z) = zc(y). \tag{1.11}$$

Now we have two possibilities : Either $a(0) \neq 0$ or $a(0) = 0$. In case $a(0) \neq \neq 0$, there is no loss of generality in assuming $a(0) = 1$, so that

$$F(y,\ z) = a(z)F(0,\ y) - zc(y) = a(y)F(0,\ z) - yc(z). \tag{1.12}$$

Substitution of this form for F in (1.1) yields

$$a(z)[a(x)\,F(0,\ y) - x\,c(y)] - a(x)[a(z)\,F(0,\ y) - zc(y)] = \\ = (x - z)\,[xzb(y) - c(y)], \tag{1.13}$$

that is to say,

$$c(y)[x - z - xa(z) + za(x)] = (x - z)xzb(y), \tag{1.14}$$

so that $g(y) = c(y)/b(y)$ is a constant, which was ruled out by hypothesis. Hence, $a(0) = 0$; and (1.11) immediately leads to (1.2). Consequently, (1.1) becomes

$$a_0zF(x,\ y) - a_0xF(y,\ z) = (x - z)\,[xzb(y) - c(y)], \tag{1.15}$$

so that

$$z[a_0F(x,\ y) - c(y) - b(y)x^2] = x\,[a_0F(y,\ z) - c(y) - b(y)z^2]. \tag{1.16}$$

Hence,

$$a_0F(x,\ y) = c(y) + d(y)x + b(y)x^2 = c(x) + d(x)y + b(x)y^2, \tag{1.17}$$

where d is some function on R_1. Since b and c are quadratic polynomials, (1.17) tells us that d is one too ; in fact, setting $y = 1$ in (1.17), we get

$$d(x) = c_1 + \theta x + b_1x^2, \tag{1.18}$$

where $\theta = d(1) - b_1 - c_1$ is arbitrary. Substitution from (1.18) into (1.17) yields (1.5).

Remark 1.2. The equation (1.1) is a simple-minded approach to the solution of Problem B in the special case where $g(x, y)$ is of the form $g(y)/x$; it is therefore not surprising that the resulting functions are all rather simple and severely limited. We are lucky that the form of g in which we are interested in the original problem belongs to the class predicated by the theorem, since $g(y) = y - \alpha = (y - \alpha)(y + 1)/(y + 1)$.

In a next Paper we shall analyse the original problem in detail through the corresponding family of orbits and give a geometric construction for solving (0.1).

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O ECUAŢIE FUNCŢIONALĂ ASOCIATĂ CU O RELAŢIE DE RECURENŢĂ NELINIARĂ

(Rezumat)

Se continuă cercetarea problemei din [1], privind natura şi şirurile care satisfac relaţia de recurenţă (0.1), studiind invaranţii asociaţi şi orbitele generate de o clasă de transformări legată de soluţionarea problemei.

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CONTRIBUŢII LA TRATAREA MATRICEALĂ PE TREI GRUPE A ECUAŢIILOR DE CONDIŢIE ÎN TRIANGULAŢIE

DE

GH. NISTOR

1. Introducere

Tratarea pe trei grupe a ecuaţiilor de condiţie după metoda K o - b î l i n, prezintă un interes deosebit în operaţia de îndesire a reţelelor de triangulaţie existente, cu puncte sau grupe de puncte noi.

Din cauza numărului mare de ecuaţii de condiţie care apar în acest caz, compensarea prin metoda tratării în bloc sau pe două grupe, devine foarte dificilă. Din această cauză, se preferă aplicarea metodei măsurărilor indirecte. Metoda tratării pe trei grupe a ecuaţiilor de condiţie permite rezolvarea, cu avantaje economice importante, a problemei, prin aplicarea metodei măsurărilor condiţionate. Ea reprezintă o generalizare a compensării reţelelor de triangulaţie prin tratarea pe două grupe a ecuaţiilor de condiţie (metoda K r ü g e r - U r m a e v).

În lucrare se prezintă o rezolvare matriceală a metodei de tratare pe trei grupe a ecuaţiilor de condiţie, prezentînd simplificări de calcul ca urmare a unei anumite reguli de numerotare a elementelor supuse compensării şi a folosirii unei matrice comune la calculul corecţiilor din etapele a II-a şi a III-a, matrice valabilă pentru toate cazurile de compensare a reţelelor cu aceeaşi configuraţie.

2. Principiul metodei

Se descompune sistemul ecuaţiilor de condiţie în trei grupe astfel ca, în grupa I-a să intre numai ecuaţiile în care necunoscutele au coeficienţi egali cu 0, $+1$, -1 şi nu se mai repetă în celelalte ecuaţii ale grupei; în grupa II-a intră ecuaţiile rămase, a căror necunoscute au coeficienţii egali cu 0, $+1$; -1; în grupa III-a intră restul ecuaţiilor.

În prima etapă, grupa I-a se tratează separat ca la metoda Krüger-Urmaev.

În etapa a doua, grupele a II-a şi a III-a, considerate împreună, se supun unei transformări ca la metoda Krüger-Urmaev, după care grupa a II-a transformată se tratează separat.

În etapa a treia, grupa III-a se supune la o nouă transformare, după care se tratează separat.

Se obţin trei valori distincte pentru fiecare corecţie, care vor trebui să satisfacă relaţia de însumare

$$V_i = \overline{V}_i + \overline{\overline{V}}_i + \overline{\overline{\overline{V}}}_i, \qquad (i = 1, 2, \ldots, n).$$

Se consideră sistemul de ecuaţii de condiţie descompus în trei grupe după criteriile arătate mai sus :

$$(1)\quad \begin{cases} \text{grupa I} & \begin{cases} a_1V_1 + a_2V_2 + \ldots + a_nV_n + W_1 = 0, \\ b_1V_1 + b_2V_2 + \ldots + b_nV_n + W_2 = 0, \\ c_1V_1 + c_2V_2 + \ldots + c_nV_n + W_3 = 0; \end{cases} \\ \text{grupa II} & \begin{cases} \alpha_1V_1 + \alpha_2V_2 + \ldots + \alpha_nV_n + W_\alpha + 0, \\ \beta_1V_1 + \beta_2V_2 + \ldots + \beta_nV_n + W_\beta = 0, \\ \gamma_1V_1 + \gamma_2V_2 + \ldots + \gamma_nV_n + W_\gamma = 0; \end{cases} \\ \text{grupa III} & \begin{cases} A_1V_1 + A_2V_2 + \ldots + A_nV_n + W_A = 0, \\ B_1V_1 + B_2V_2 + \ldots + B_nV_n + W_B = 0. \end{cases} \end{cases}$$

Ecuaţiile matriceale corespunzătoare sînt :

$$(2)\quad \begin{cases} UV + W_s = 0, \\ \alpha V + W_\alpha = 0, \\ AV + W_A = 0. \end{cases}$$

Rezolvînd aceste ecuaţii sub condiţia

$$(3)\qquad V^*EV = \min.$$

se ajunge la ecuaţia normală a corelatelor

$$(4)\qquad NK + W = 0,$$

a cărei matrice dezvoltată este

$$
(5)\quad
\left(\begin{array}{ccc|ccc|cc|c}
n_1 & 0 & 0 & \sum\limits_1^{n_1}\alpha & \sum\limits_1^{n_1}\beta & \sum\limits_1^{n_1}\gamma & \sum\limits_1^{n_1}A & \sum\limits_1^{n_1}B & w_1 \\
0 & n_2 & 0 & \sum\limits_{n_1+1}^{n_2}\alpha & \sum\limits_{n_1+1}^{n_2}\beta & \sum\limits_{n_1+1}^{n_2}\gamma & \sum\limits_{n_1+1}^{n_2}A & \sum\limits_{n_1+1}^{n_2}B & w_2 \\
0 & 0 & n_3 & \sum\limits_{n_2+1}^{n_3}\alpha & \sum\limits_{n_2+1}^{n_3}\beta & \sum\limits_{n_2+1}^{n_3}\gamma & \sum\limits_{n_2+1}^{n_3}A & \sum\limits_{n_2+1}^{n_3}B & w_3 \\
\hline
\sum\limits_1^{n_1}\alpha & \sum\limits_{n_1+1}^{n_2}\alpha & \sum\limits_{n_2+1}^{n_3}\alpha & [\alpha\alpha] & [\alpha\beta] & [\alpha\gamma] & [\alpha A] & [\alpha B] & w \\
\sum\limits_1^{n_1}\beta & \sum\limits_{n_1+1}^{n_2}\beta & \sum\limits_{n_2+1}^{n_3}\beta & [\alpha\beta] & [\beta\beta] & [\beta\gamma] & [\beta A] & [\beta B] & w_\beta \\
\sum\limits_1^{n_1}\gamma & \sum\limits_{n_1+1}^{n_2}\gamma & \sum\limits_{n_2+1}^{n_3}\gamma & [\beta\gamma] & [\beta\gamma] & [\gamma\gamma] & [\gamma A] & [\gamma B] & w_\gamma \\
\hline
\sum\limits_1^{n_1}A & \sum\limits_{n_1+1}^{n_2}A & \sum\limits_{n_2+1}^{n_3}A & [\alpha A] & [\beta A] & [\gamma A] & [AA] & [AB] & w_A \\
\sum\limits_1^{n_1}B & \sum\limits_{n_1+1}^{n_2}B & \sum\limits_{n_2+1}^{n_3}B & [\alpha B] & [\beta B] & [\gamma B] & [AB] & [BB] & w_B
\end{array}\right).
$$

Prof. Mazmişvili a arătat că, folosind matricele, metodele de tratare pe grupe a ecuaţiilor de condiţie pierd orice deosebire, ele reprezentînd transformarea unei matrice bloc într-o matrice cvazitriunghiulară. În cazul compensării pe trei grupe, problema se reduce la transformarea matricei bloc, obţinută din matricea dezvoltată (5), într-o matrice cvazitriunghiulară

$$
(6)\quad
\begin{pmatrix} n_s & \alpha^* & A^* & w_s \\ \alpha & \alpha\alpha^* & \alpha A^* & w_\alpha \\ A & \alpha^* A & AA^* & w_A \end{pmatrix}
\rightarrow
\begin{pmatrix} n_s & \alpha^* & A^* & w_s \\ & \overline{\alpha\alpha}^* & \overline{\alpha A}^* & \overline{w}_\alpha \\ & & \overline{\overline{AA}}^* & \overline{\overline{w}}_A \end{pmatrix}.
$$

Iniţial se calculează doar elementele submatricelor n_s, α^*, A^*, operaţie extrem de simplă, după cum se vede în (5), respectîndu-se următoarea regulă : numerotarea elementelor supuse compensării se va face astfel încît, în ecuaţiile grupei I-a, necunoscutele să se prezinte în ordine succesivă de la 1 la n.

În prima etapă, tratîndu-se grupa I-a separat, înseamnă că neînchiderile se repartizează în mod egal la toate necunoscutele care intră în ecuaţiile grupei respective.

Din rezolvarea primelor ecuaţii $n_s K_s + W_s = 0$ se obţine $K_s = -\frac{W_s}{n_s}$ unde: $s = 1, 2, 3$; W_s — termenul liber al ecuaţiei de condiţie a grupei I-a, cu numărul de ordine s; n_s — numărul necunoscutelor din ecuaţia de condiţie a grupei I-a, cu numărul de ordine s.

Din ecuaţia care exprimă corecţiile funcţie de corelate $\bar{V}_i = K_s$, se obţin valorile corecţiilor din prima etapă

$$\bar{V}_{is} = -\frac{W_s}{n_s}, \quad (i = 1, 2, \ldots, n). \tag{7}$$

În etapa a doua, se transformă ecuaţiile din grupele II şi III, considerate împreună, ca la metoda Krüger-Urmaev. Aceasta se face supunîndu-se matricea bloc, obţinută din (5), unei transformări triunghiulare Gauss-Doolittle, respectiv

$$\begin{pmatrix} n_s & \alpha^* & A^* & w_s \\ \alpha & \alpha\alpha^* & \alpha A^* & w_\alpha \\ A & \alpha^* A & AA^* & w_A \end{pmatrix} \rightarrow \begin{pmatrix} n_s & \alpha^* & A^* & w_s \\ & \overline{\alpha\alpha^*} & \overline{\alpha A^*} & \overline{w_\alpha} \\ & \overline{\alpha^* A} & \overline{AA^*} & \overline{w_A} \end{pmatrix}. \tag{8}$$

Operatorul de transformare ortogonală se exprimă prin formula

$$\rho = -\frac{1}{n_s}\alpha^* = -\frac{1}{n_s}A^*, \tag{9}$$

a cărui elemente (corelatele auxiliare) sînt:

$$\rho_{11} = -\frac{1}{n_1}\sum_{1}^{n_1}\alpha, \qquad \rho_{21} = -\frac{1}{n_1}\sum_{1}^{n_1}\beta, \qquad \rho_{31} = -\frac{1}{n_1}\sum_{1}^{n_1}\gamma,$$

$$\rho_{12} = -\frac{1}{n_2}\sum_{n_1+1}^{n_2}\alpha, \qquad \rho_{22} = -\frac{1}{n_2}\sum_{n_1+1}^{n_2}\beta, \qquad \rho_{32} = -\frac{1}{n_2}\sum_{n_1+1}^{n_2}\gamma,$$

$$\rho_{13} = -\frac{1}{n_3}\sum_{n_2+1}^{n_3}\alpha, \qquad \rho_{23} = -\frac{1}{n_3}\sum_{n_2+1}^{n_3}\beta, \qquad \rho_{33} = -\frac{1}{n_3}\sum_{n_2+1}^{n_3}\gamma,$$

$$\rho_{41} = -\frac{1}{n_1}\sum_{1}^{n_1}A, \qquad \rho_{51} = -\frac{1}{n_1}\sum_{1}^{n_1}B,$$

$$\rho_{42} = -\frac{1}{n_2}\sum_{n_1+1}^{n_2}A, \qquad \rho_{52} = -\frac{1}{n_2}\sum_{n_1+1}^{n_2}B,$$

$$\rho_{43} = -\frac{1}{n_3}\sum_{n_2+1}^{n_3}A, \qquad \rho_{53} = -\frac{1}{n_3}\sum_{n_2+1}^{n_3}B.$$

Cu ajutorul corelatelor auxiliare se calculează coeficienţii şi termenii liberi ai grupelor a II-a şi a III-a transformate, ca la metoda Krüger-Urmaev, cu relaţiile

$$\bar{\alpha}_{is} = \alpha_{is} - \rho_{1s}\,; \quad \bar{\beta}_{is} = \beta_{is} - \rho_{2s}\,; \quad \bar{\gamma}_{is} = \gamma_{is} - \rho_{3s}$$
$$\bar{A}_{is} = A_{is} - \rho_{4s}\,; \quad \bar{B}_{is} = B_{is} - \rho_{5s} \tag{10}$$

unde

$$i = 1,\ 2\ ,\ldots.\ n, \qquad s = 1,\ 2,\ 3.$$

Pentru simplificarea calculelor, termenii liberi ai grupei a II-a, vor fi deduşi din mărimile afectate cu primele părţi ale corecţiilor ($\bar{V}$). În acest fel termenii liberi nu mai trebuie transformaţi, astfel că $\bar{W}_\alpha = W_\alpha$, $\bar{W}_\beta = W_\beta$, $\bar{W}_\gamma = W_\gamma$.

Ecuaţiile matriceale ale grupelor a II-a şi a III-a transformate vor fi

$$\bar{\alpha}V + W_\alpha = 0, \tag{11}$$

$$\bar{A}V + W_A = 0. \tag{12}$$

În continuare, se tratează independent grupa a II-a transformată. Ecuaţia matriceală corespunzătoare sistemului de ecuaţii normale a corelatelor, a grupei a II-a transformate, este

$$\bar{\alpha}\bar{\alpha}^* K_\alpha + W_\alpha = 0 \quad \text{sau} \quad \bar{N}_2 K_\alpha + W_\alpha = 0, \tag{13}$$

de unde, vectorul corelatelor va fi

$$K_\alpha = -\bar{N}_2^{-1} W_\alpha = \bar{Q}_2 W_\alpha. \tag{14}$$

Corecţiile din etapa a doua se determină după formulele de la metoda corelatelor

$$\bar{\bar{V}} = \bar{\alpha}^* K_\alpha = \bar{\alpha}^* \bar{Q}_2 W_\alpha \tag{15}$$

sau

$$\begin{pmatrix} \bar{\bar{V}}_1 \\ \bar{\bar{V}}_2 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ V_n \end{pmatrix} = \begin{pmatrix} \bar{\alpha}_1 & \bar{\beta}_1 & \bar{\gamma}_1 \\ \bar{\alpha}_2 & \bar{\beta}_2 & \bar{\gamma}_2 \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \bar{\alpha}_n & \bar{\beta}_n & \bar{\gamma}_n \end{pmatrix} \begin{pmatrix} \bar{Q}_{11} & \bar{Q}_{21} & \bar{Q}_{31} \\ \bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{32} \\ \bar{Q}_{13} & \bar{Q}_{23} & \bar{Q}_{33} \end{pmatrix} \begin{pmatrix} w_\alpha \\ w_\beta \\ w_\gamma \end{pmatrix}.$$

Notînd $\bar{\alpha}^*\bar{Q}_2 = R$, se obţine $\bar{\bar{V}} = RW_\alpha$ sau

$$(16)\qquad \begin{pmatrix} \bar{\bar{V}}_1 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \bar{\bar{V}}_n \end{pmatrix} = \begin{pmatrix} p_1 & q_1 & r_1 \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ p_n & q_n & r_n \end{pmatrix} \begin{pmatrix} w_\alpha \\ w_\beta \\ w_\gamma \end{pmatrix}.$$

Se observă că practic, pentru calculul corecţiilor secunde, nici nu este nevoie să se calculeze vectorul corelatelor, ci doar matricea inversă $\bar{Q}_2$, care intră în calculul elementelor matricei R.

În etapa a treia se procedează la transformarea, a doua oară, a coeficienţilor necunoscutelor din grupa a III-a, după metoda Krüger-Urmaev. Şi în acest caz, problema se reduce la a aduce matricea coeficienţilor ecuaţiilor normale ale grupelor a II-a şi a III-a, obţinute din prima transformare, la o formă cvazitriunghiulară.

$$(17)\qquad \begin{pmatrix} \bar{\alpha}\bar{\alpha}^* & \bar{\alpha}\bar{A}^* & \bar{w}_\alpha \\ \bar{\alpha}^*\bar{A} & \bar{A}\bar{A}^* & \bar{w}_A \end{pmatrix} \rightarrow \begin{pmatrix} \alpha\alpha^* & \bar{\alpha}\bar{A}^* & \bar{w}_\beta \\ & \bar{\bar{A}}\bar{\bar{A}}^* & \bar{\bar{w}}_A \end{pmatrix}.$$

Operatorul de transformare ortogonală este

$$(18)\qquad \rho^0 = -\bar{N}_2^{-1}\bar{\alpha}\bar{A}^* = \bar{Q}_2\bar{\alpha}\bar{A}^*,$$

coeficienţii necunoscutelor din ecuaţiile grupei a III-a, transformaţi a doua oară, vor fi

$$(19)\qquad \bar{\bar{A}} = \bar{A} + \bar{\alpha}^*\rho^0 = \bar{A} + \bar{\alpha}^*\bar{Q}_2\bar{\alpha}\bar{A}^* = \bar{A} + R\bar{\alpha}\bar{A}^*.$$

Dar, ca şi în cazul metodei Krüger-Urmaev aplicată la două grupe, se obţine

$$\bar{\alpha}\bar{A}^* = \alpha\bar{A}^* = T = \begin{pmatrix} \Sigma\bar{A}_\alpha & \Sigma\bar{B}_\alpha \\ \Sigma\bar{A}_\beta & \Sigma\bar{B}_\beta \\ \Sigma\bar{A}_\gamma & \Sigma\bar{B}_\gamma \end{pmatrix},$$

unde $\Sigma\bar{A}_\alpha, \ldots, \Sigma\bar{B}_\alpha \ldots$ reprezintă suma coeficienţilor $\bar{A}$ şi $\bar{B}$ de indici egali cu cei ai necunoscutelor existente în ecuaţiile 1, 2, 3 ai grupei a II-a. Coeficienţii necunoscutelor din ecuaţiile grupei a III-a, transformaţi a doua oară, vor fi

(20) $$\bar{\bar{A}} = \bar{A} + RT.$$

Şi în această etapă, termenii liberi nu suferă transformări deoarece ei se calculează după aplicarea corecţiilor $\bar{V}$ şi $\bar{\bar{V}}$ mărimilor ce se compensează. Ecuaţia matriceală a grupei a III-a capătă forma

(21) $$\bar{\bar{A}}V + W_A = 0.$$

Ecuaţia matriceală, corespunzătoare sistemului de ecuaţii normale a corelatelor, a grupei a III-a, este

(22) $$\bar{\bar{A}}\bar{\bar{A}}^* K_A + W_A = 0, \quad \text{sau} \quad \bar{\bar{N}}_3 K_A + W_A = 0.$$

Prin rezolvare se obţine vectorul corelatelor

(23) $$K_A = -\bar{\bar{N}}_3^{-1} W_A = \bar{\bar{Q}}_3 W_A .$$

Se calculează corecţiile, din a treia etapă, cu formulele de la metoda corelatelor

(24) $$\bar{\bar{\bar{V}}} = \bar{\bar{A}}^* K_A.$$

În final, corecţiile totale ce se aplică mărimilor compensate sînt date de vectorul

(25) $$V = \bar{V} + \bar{\bar{V}} + \bar{\bar{\bar{V}}}.$$

3. Concluzii

1) Tratarea matriceală pe trei grupe a ecuaţiilor de condiţie, permite o înţelegere uşoară a metodei clasice de rezolvare a acestei probleme.

2) Ea conduce la o simplitate a calculului în toate etapele de compensare.

3) Aplicarea acestei metode, în cazul folosirii măsurărilor condiţionate la îndesirea punctelor reţelei de triangulaţie, face ca numărul mare de ecuaţii de condiţie care apar, să nu prezinte niciun inconvenient.

4) Folosirea matricelor permite scrierea comodă a unui program de rezolvare a acestei probleme, la maşinile electronice de calcul.

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şi fotogrametrie

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О РАЗДЕЛЬНОМ МАТРИЧННОМ РЕШЕНИИ ПО ТРЕХ ГРУППАМ УСЛОВНЫХ УРАВНЕНИЙ ТРИАНГУЛЯЦИЙ

(Резюме)

В работе даётся матричное раздельное решение по группам условных уравнений представляя расчетные упрощения и элементы действительные для одинаковых сетей.

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TOMUL XX (XXIV), fasc. 1—2, 1974
Secţia I
MATEMATICĂ
MECANICĂ TEORETICĂ, FIZICĂ

C.Z. 515.69

DEMONSTRAREA TEOREMEI POHLKE—SCHWARTZ PRIN RAŢIONAMENTE ÎN EPURĂ

DE

AL. MATEI, TATIANA TACU şi VICTOR GABA

1. Introducere

Problema reprezentărilor axonometrice a atras mulţi cercetători. Printre aceştia se numără şi geometrul Pohlke, care a enunţat teorema principală a axonometriei paralele, care îi poartă numele şi anume: *oricare trei segmente care pleacă dintr-un punct în plan, pot fi considerate ca proiecţia paralelă a trei segmente egale şi reciproc perpendiculare în spaţiu.*

Demonstraţia acestei teoreme era foarte greoaie, în timp ce conţinutul geometric al ei, este relativ simplu. Aceasta a determinat să se caute o demonstraţie mai simplă.

O astfel de demonstraţie, a fost dată nu peste mult timp de Schwartz, care plecînd de la cele realizate deja a fixat pe axele unui triedru dreptunghic, începînd din origine, segmente egale între ele. Unind apoi extremităţile acestora a obţinut un tetraedru, pe care l-a proiectat după o direcţie arbitrară pe un plan oarecare şi a obţinut ca imagine a lui un patrulater complet, adică şi cele două diagonale. Acest patrulater poate să aibă orice formă dinainte dată, şi să se aleagă direcţia de proiectare în aşa fel ca el să reprezinte tetraedrul din spaţiu cu unghiurile dintr-un vîrf drepte. De aici a apărut teorma generalizată Pohlke-Schwartz: *oricare patrulater în plan, nedeformat complet, poate fi considerat ca o proiecţie paralelă a unui tetraedru asemenea cu oricare tetraedru dat.*

2. Demonstraţia teoremei

Fie pe fig. 1 *a* dat în plan patrulaterul $ABCD$ şi pe fig. 1 *b*, dat în epură tetraedrul $A_1B_1C_1D_1$ $(a_1b_1c_1d_1,\ a_1'b_1'c_1'd_1')$. Se consideră patrulaterul dat proiecţia unui tetraedru oarecare, deci şi a tetraedrului dat. Cele şase

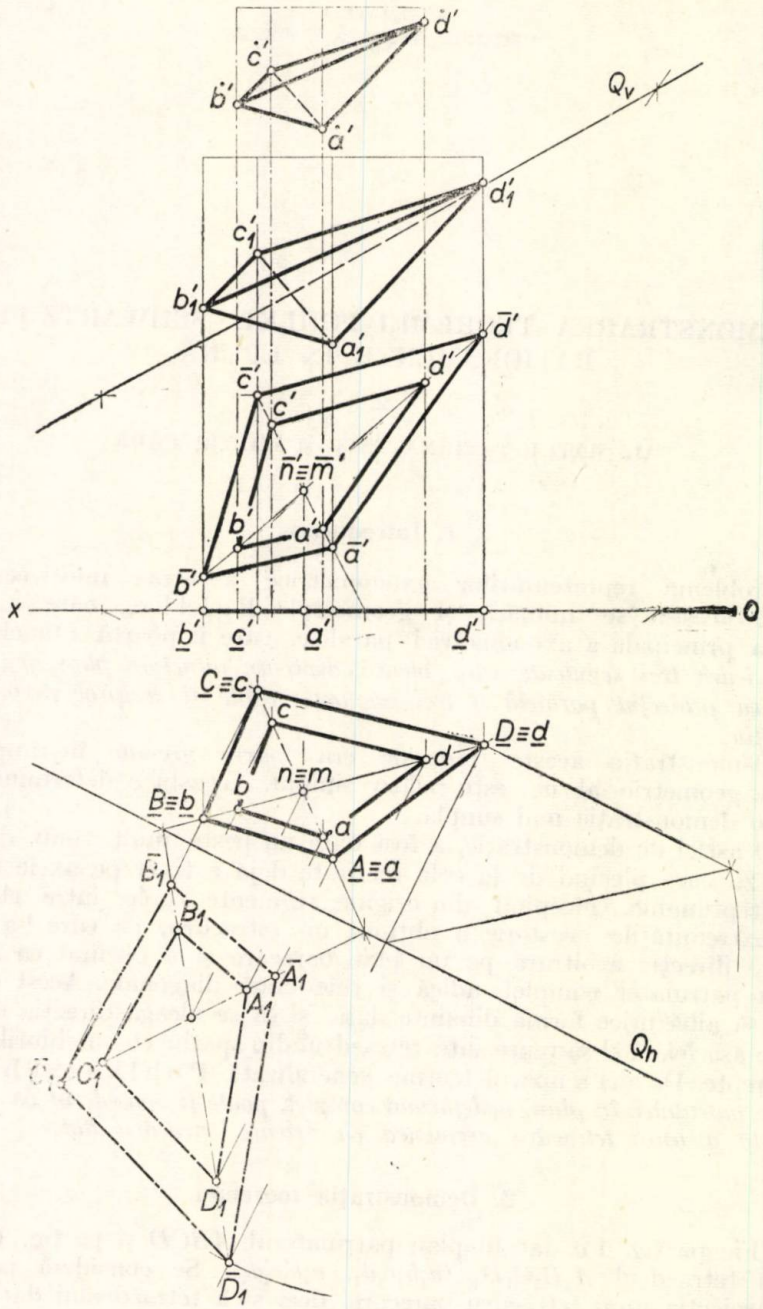
d'
c'
b'
a'
Qv
d'1
c'1
b'1
a'1
d̄'
c̄'
d''
c''
n̄'≡m̄'
b''
a''
ā'
b̄'
x
0
b̄'
c̄'
ā'
d̄'
C≡c
c
D≡d
d
n≡m
b
B≡b
a
A≡a
B̄1
B1
Ā1
A1
Qh
C̄1
C1
D1
D̄1

Fig. 1

laturi ale patrulaterului complet sînt proiecţiile muchiilor tetraedrului. Referindu-ne numai la diagonalele patrulaterului, ele reprezintă proiecţiile a două muchii ale tetraedrului, adică în cazul fig. 1 *b*, a muchiilor A_1C_1 şi B_1D_1, pe care se pot indica punctele M_1 şi N_1, ale căror proiecţii se găsesc la intersecţia diagonalelor patrulaterului, adică în M şi N. Cum patrulaterul este dat, avînd în vedere raportul simplu a trei puncte, se pot găsi exact punctele $M_1(m_1, m_1')$ şi $N_1(n_1, n_1')$ din rapoartele:

$$(1) \qquad \frac{AM}{MC} = \frac{a_1'm_1'}{m_1'c_1'} = \frac{a_1m_1}{m_1c_1} \quad \text{şi} \quad \frac{BN}{ND} = \frac{b_1'n_1'}{n_1'd_1'} = \frac{b_1n_1}{n_1d_1}.$$

Se duce apoi prin punctele găsite dreapta (Δ) $[(\delta), (\delta')]$, direcţia de proiectare. Avînd această direcţie şi fiind vorba de proiecţia paralelă, se duc prin toate vîrfurile tetraedrului, proiectante, care se pot considera ca muchiile unei prisme.

Secţionînd prisma imaginată cu un plan P (fig. 1 *b*) perpendicular pe (Δ), se obţine patrulaterul $\underline{ABCD}$ $(\underline{abcd}, \underline{a'b'c'd'})$, care este proiecţia ortogonală a tetraedrului pe acest plan şi secţiunea dreaptă în prismă.

În baza păstrării raportului simplu a trei puncte avem:

$$(2) \qquad \frac{a_1'm_1'}{m_1'c_1'} = \frac{\underline{a'}\,\underline{m'}}{\underline{m'}\,\underline{c'}}, \quad \frac{b_1'n_1'}{n_1'd_1'} = \frac{\underline{b'}\,\underline{n'}}{\underline{n'}\,\underline{d'}}, \quad \text{şi} \quad \frac{a_1m_1}{m_1c_1} = \frac{\underline{a}\,\underline{m}}{\underline{m}\,\underline{c}}, \quad \frac{b_1n_1}{n_1d_1} = \frac{\underline{bn}}{\underline{nd}}.$$

Din (1) şi (2) se obţine:

$$(3) \qquad \frac{\underline{a'}\,\underline{m'}}{\underline{m'}\,\underline{c'}} = \frac{\underline{am}}{\underline{mc}} = \frac{AM}{MC} \quad \text{şi} \quad \frac{\underline{b'}\,\underline{n'}}{\underline{n'}\,\underline{d'}} = \frac{\underline{bn}}{\underline{nd}} = \frac{BN}{ND}.$$

Deci patrulaterele $ABCD$ şi $\underline{ABCD}$ $(\underline{abcd}, \underline{a'b'c'd'})$ pot fi aduse în corespondenţă perspectiv afină şi apoi în poziţie perspectiv ortogonală.

În acest scop se determină mai întîi mărimea adevărată a patrulaterului $\underline{ABCD}$. Pe urmă, din împărţirea patrulaterelor, se consideră două triunghiuri corespondente, de exemplu triunghiurile $\underline{ABC}$ şi ABC, pe care aducîndu-le în corespondenţă afină, făcînd ca latura $\underline{AC}$ să fie egală cu AB etc. şi apoi folosind direcţiile principale, ajung în corespondenţă perspectiv ortogonală.

Din aceste operaţiuni a rezultat patrulaterul $\overline{ABCD}$ (fig. 1 *a*), asemenea cu patrulaterul $ABCD$ şi afin cu patrulaterul $\underline{ABCD}$ (fig. 1 *b*) care este proiecţia lui ortogonală.

Pe fig. 2, luînd în planul orizontal patrulaterul $\underline{ABCD}$ (secţiunea dreaptă) şi ducînd muchiile prismei considerate, se pot lua pe acestea vîrfurile patrulaterului $\overline{ABCD}$, care reprezintă secţiunea prismei cu un plan oarecare.

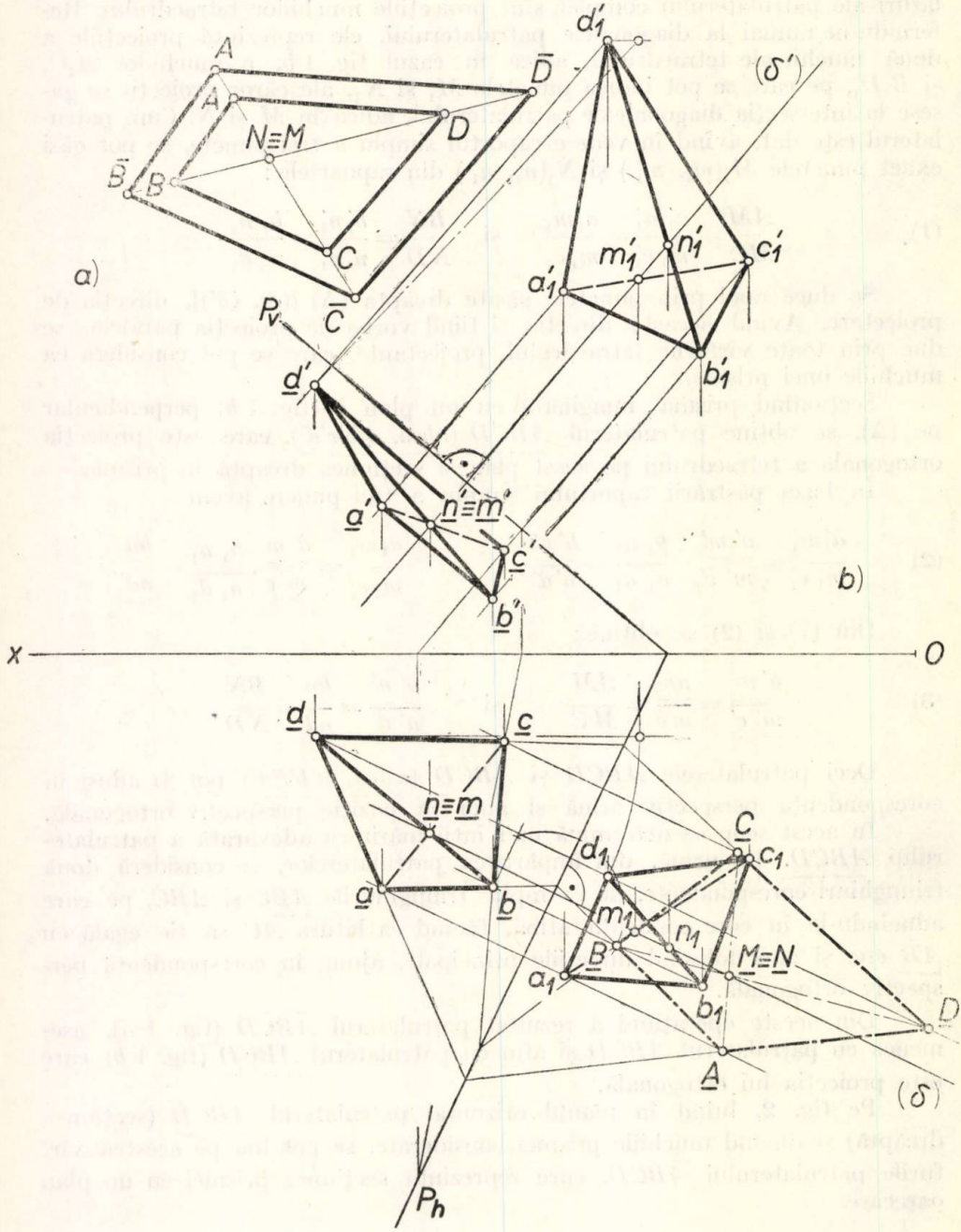
A
D
A
N≡M
D
B
B
C
a)
C
P_v
d_1'
(δ')
m_1'
n_1'
c_1'
a_1'
b_1'
d'
a'
n'≡m'
c'
b'
b)
x
O
d
c
n≡m
C
c_1
d_1
a
b
m_1
n_1
B
M≡N
a_1
b_1
D
A
(δ)
P_h

Fig. 2

Pe figură este indicată numai proiecţia verticală $\bar{a}'\ \bar{b}'\ \bar{c}'\ \bar{d}'$, proiecţia orizontală este identică cu $\overline{ABCD} \equiv \overline{abcd}$.

Rămîne acum ca pe fig. 2 să se reprezinte şi tetraedrul $A_1B_1C_1D_1$ a cărui proiecţie orizontală (care nu este indicată), va fi şi ea identică cu $\overline{abcd}$. În ce priveşte proiecţia verticală, trebuie să se afle mărimea adevărată a porţiunilor din muchiile prismei proiectante de pe fig. 1 *b*, adică mărimea adevărată a segmentelor AA_1 (aa_1, $a'a_1'$), BB_1 (bb_1, $b'b_1'$), CC_1 (cc_1, $c'\ c_1'$) şi DD_1 (dd_1, d', d_1'), care se instalează apoi pe muchiile corespunzătoare în fig. 2.

Astfel se obţine şi tetraedrul $A_1B_1C_1D_1$ (fig. 2), ale cărui proiecţii sînt cele două patrulatere şi deci patrulaterul $\overline{ABCD}$ ($\overline{abcd}$, $\overline{a'b'c'd'}$) este proiecţia paralelă a tetraedrului $A_1B_1C_1D_1$ ($a_1b_1c_1d_1$, $a_1'b_1'c_1'd_1'$).

Pînă aici patrulaterul $ABCD$ nu apare ca proiecţia unui tetraedru ; în baza celor de mai înainte, continuînd raţionamentul, se poate arăta însă şi acest aspect.

Întîi se determină urmele planului Q în care se găseşte patrulaterul $\overline{ABCD}$ (fig. 2) şi se aduce patrulaterul în planul orizontal de proiecţie în $\bar{A}_1\bar{B}_1\bar{C}_1\bar{D}_1$ unde este trasat cu linie punct.

Se trasează aici şi patrulaterul $ABCD$, care este asemenea cu precedentul şi aparţine planului Q, notat fiind cu $A_1B_1C_1D_1$. Procedînd apoi la găsirea afinului lui, se obţine proiecţia orizontală $abcd$ şi prin ridicarea planului şi proiecţia verticală $a'b'c'd'$, care sînt respectiv secţiunea dreaptă (care nu este notată) şi secţiunea cu acelaşi plan Q printr-o prismă dreaptă, ale cărei muchii laterale sînt trasate cu linie punct subţire.

Prisma proiectantă poate fi tăiată cu o infinitate de plane, fiecare patrulater rezultat putînd fi considerat ca proiecţia unui tetraedru. Dacă se consideră din infinitatea de patrulatere numai unul, pe muchiile prismei imaginate, se pot lua pe fiecare cîte o infinitate de puncte.

Considerînd apoi cîte un punct pe fiecare muchie (se exclude cazul cînd cele patru puncte ar fi în acelaşi plan) şi unindu-le între ele, se obţine un tetraedru ; în mod analog se pot obţine o infinitate de tetraedre, care se proiectează toate după patrulaterul considerat $ABCD$. În infinitatea de tetraedre este conţinut şi unul cu unghiul triedric drept într-un vîrf ; pe laturile unghiurilor drepte, tetraedrul va avea segmente egale.

Rezultă astfel teorema lui Pohlke ca un caz particular al teoremei generalizate. Anume, dacă se unesc extremităţile celor trei segmente indicate de Pohlke, duse din acelaşi punct în plan, se obţine patrulaterul care poate fi considerat proiecţia paralelă a tetraedrului cu un unghi triedru drept şi cu muchiile care sînt laturile unghiurilor drepte, egale.

În fig. 2 s-a trasat tetraedrul $\dot{A}\dot{B}\dot{C}\dot{D}$ ($\dot{a}\dot{b}\dot{c}\dot{d}$, $\dot{a}'\dot{b}'\dot{c}'\dot{d}'$), care este asemenea cu tetraedrul $A_1B_1C_1D_1$ şi are ca proiecţie patrulaterul dat $ABCD$ (nu este indicată proiecţia orizontală $\dot{a}\dot{b}\dot{c}\dot{d}$, ea este însă identică cu $abcd$).

Astfel teorma lui Pohlke este complet justificată cu mijloacele geometriei descriptive.

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ÉTABLISSEMENT DU THÉORÈME DE POHLKE—SCHWARTZ EN RAISONNANT DANS L'ÉPURE

(Résumé)

On présente le théorème de Pohlke, puis celui de Pohlke et Schwartz en faisant tout le raisonnement à l'aide des épures de la géométrie descriptive.

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Secţia I
MATEMATICĂ
MECANICĂ TEORETICĂ, FIZICĂ

C.Z. 515.3

GEOMETRIA REPREZENTATIVĂ DE CORESPONDENŢĂ ÎN PROIECŢIA CILINDRICĂ OBLICĂ [1])

DE

I. RUSU

1. Introducere

Considerînd ca sistem de referinţă diedrul variabil, se construieşte proiecţia cilindrică pe unul din planele sistemului, utilizînd în acest scop proiecţia ortogonală a obiectului dat în spaţiu pe celălalt plan, precum şi corespondenţa ce se poate stabili între cele două proiecţii.

2. Proiecţia punctului

Se consideră planele $[H]$ şi $[P]$, a căror dreaptă de intersecţie este (Δ). Se stabileşte prin convenţie că planul $[H]$ este un plan orizontal de cotă zero, pe care un element geometric sau obiect se reprezintă prin proiecţia sa ortogonală. Planul $[P]$ este considerat plan de proiecţie şi determinat de dreapta (Δ) şi de un punct A $(a\,;\,Z_A)$, fig. 1.

Fie dreapta (D), proiectanta punctului $M(m\,;\,0)$ determinată şi de L $(l\,;\,Z_L)$. Se poate admite că: $M = (D) \cap [H]$: deci acest punct este urma pe planul $[H]$ al reprezentării a dreptei (D).

Dar $(D) \cap [P] = C$, ceea ce înseamnă că proiecţia punctului M pe planul $[P]$ este punctul C şi se numeşte oblică pentru că dreapta considerată (D) face un unghi diferit de $90°$ cu planul proiecţiei.

Dreapta (D) este direcţia de proiectare, planele $[H]$ şi $[P]$ sînt planele corespondente a căror dreaptă comună (Δ) este axă de omologie, în cazul de faţă axă de afinitate. Punctele M şi C ce-şi corespund unul altuia sînt

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[1]) Prezentată la sesiunea ştiinţifică a Institutului de Construcţii din Bucureşti, 24—26 aprilie 1969.

unul omologul afin al celuilalt şi deci corespondenţa de omologie afină este determinată, astfel că unei figuri din planul $[H]$ îi corespunde o figură afină din planul $[P]$ şi reciproc, figură care poate fi construită.

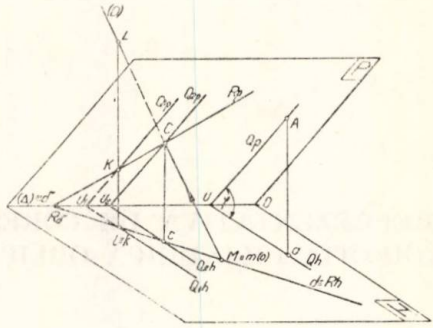

Fig. 1

În cazul de faţă s-a considerat prin dreapta (D), planul $[R]$ perpendicular pe planul $[H]$, deci un plan vertical. Astfel: $(Rh) \equiv (d) = [R] \cap \cap [H]$ = urmă pe planul $[H]$ a planului $[R]$; $(Rp) = [R] \cap [P]$ = urmă pe planul $[P]$ a planului $[R]$.

Pentru găsirea urmei (Rp) este necesar să i se cunoască două puncte astfel: unul din ele R rezultă din intersecţia urmei (Rh) cu axa (Δ). Al doilea punct se găseşte, spre exemplu la intersecţia verticalei punctului L cu planul $[P]$.

Prin urmare $K = (Ll) \cap [P]$. Unind prin linie dreaptă punctele R şi K se obţine urma (Rp), ca dreaptă de intersecţie a planului proiectant $[R]$ cu planul de proiecţie $[P]$ şi care intersectează dreapta (D) considerată în punctul căutat C. Punctul K s-a găsit considerîndu-se plauul $(Q_1) \perp (\Delta)$ dus prin verticala (Ll). Urma (Q_1h) trece prin $l \equiv K$ şi este perpendiculară pe (Δ) sau $(Q_1h) \parallel (Qh)$. De asemenea urma $(Q_{1p}) \equiv (Q_p)$ pentru că urmele (Qh), (Q_1h) (Qp), (Q_1p), rezultă din intersecţia planelor $[Q]$ şi $[Q_1]$ paralele cu planele corespondente $[H]$ şi respectiv $[P]$.

Trecerea la epură se face prin rabaterea planelor $[R]$, $[Q]$ şi $[Q_1]$ împreună cu elementele conţinute pe planul reprezentării $[H]$. Astfel, planul $[P]$ este determinat de dreapta (D) şi de punctul $A(a\,;\,3)$, fig 2. Se consideră apoi planul $[Q]$ prin punctul $A\,(a\,;\,3)$ perpendicular pe axa (Δ) astfel că urma (Qh) trece prin a şi este perpendiculară pe (Δ). Considerînd această urmă axă de rabatere, se duce prin a perpendiculara la (Qh), pe care se măsoară din a, $Z_A = 3$ unităţi, găsindu-se astfel punctul A rabătut pe planul reprezentării $[H]$.

Din cele arătate rezultă că urma (Qp), ca intersecţie dintre planul $[Q]$ şi planul de proiecţie $[P]$, este cea de a doua latură a unghiului

plan corespunzător, deci unghiul $AUa = \varphi$. Se rabate dreapta (D) în poziţia (D_0) pe planul $[H]$ rabatînd punctele ei, $(d) \equiv (Rh)$ fiind axă de rabatere. Deci din $l(3, 5)$ se duce o perpendiculară la axa $(Rh) \equiv (d)$ şi se măsoară cota $Z_L = 3{,}5$ unităţi. Punctul $M \equiv m$, care este şi propriul său rabătut, se uneşte printr-o dreaptă cu L, obţinîndu-se dreapta (D_0) rabătută. Se găseşte cota punctului K, prin construirea ΔKU_1L, ce are unghiul din U_1 egal cu unghiul din U al ΔAUa.

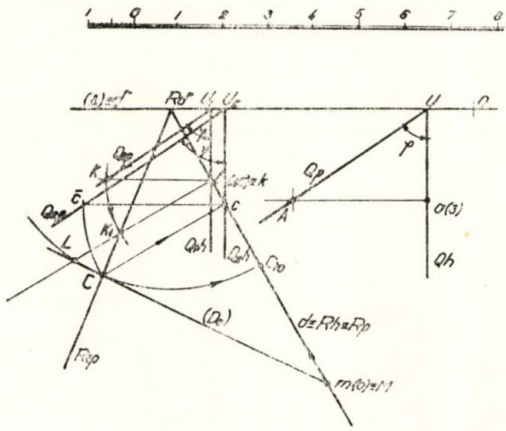

Fig. 2

Se măsoară cota $Z_k = kK$ pe perpendiculara dusă la $(Rh) \equiv (d)$ prin $l \equiv k$, prin trasarea arcului de cerc cu centrul în $l \equiv k$ şi de rază egală cu $Z_k = kK = kK_1$. Unind printr-o dreaptă punctele R_δ şi K_1 se obţine urma (R_1p) rabătută în planul $[H]$. Acesta se intersectează cu (D_0) în C, care proiectat ortogonal pe $(d) \equiv (Rh)$ conduce la obţinerea proiecţiei c. Deci $C(c\,;\,Cc) \in D(M\,;\,L)$.

Acest punct găsit în epură la intersecţia dintre (D_0) şi (R_1p) poate fi şi rabătut pe planul reprezentării $[H]$, rabătînd urma (R_1p), astfel încît să rămînă tot timpul în planul vertical $[R]$, pînă ce urma (R_1p) se confundă cu urma (Rh) a aceluiaşi plan. Punctul C de intersecţie dintre (R_1p) şi (D_0) descrie în planul $[R]$ un cerc cu centrul în R_δ şi de rază $R_\delta C$, cerc care intersectează urma (Rh), deci planul reprezentării $[H]$ în punctele C_{10} şi C_{20}, care sînt poziţiile rabătute ale punctului C. Pe epură s-a reprezentat numai poziţia rabătută C_{10}, ce reprezintă una din soluţiile găsite.

În acest mod corespondenţa de omologie afină dintre planele $[H]$ şi $[P]$ este determinată.

3. Proiecţiile afine ale unor figuri plane

3.1. Proiecţia unui pătrat

Fie un pătrat situat în planul reprezentării $[H]$, direcţia de proiectare fiind dreapta (D) $(d\,;\,D_0)$ de poziţie generală faţă de planul de proiecţie $[P]$. Această dreaptă este determinată de punctele $A\,(a\,;\,0)$, care este unul din vîrfurile pătratului $ABCE$ dat şi $L\,(l\,;\,3{,}5)$, un punct al dreptei D.

Planul de proiecţie $[P]$ este determinat de aceeaşi dreaptă $(\Delta) \equiv (\delta)$, fig. 3 şi de punctul $M(m\,;\,2)$. Prin punctul M se duce planul $[Q] \perp (\Delta)$, astfel că: $m \in (Qh) \perp (\Delta)$. Urma (Qp) a acestui plan se rabate în planul $[H]$ al reprezentării, rabatînd punctul M, axă de rabatere fiind urma (Qh). De aceea se duce prin m o perpendiculară la (Qh) pe care se măsoară cota acestui punct astfel că $Z_M = mM = 2$ unităţi de pe scara grafică adoptată Deci $(UM) \equiv (Q_p)$.

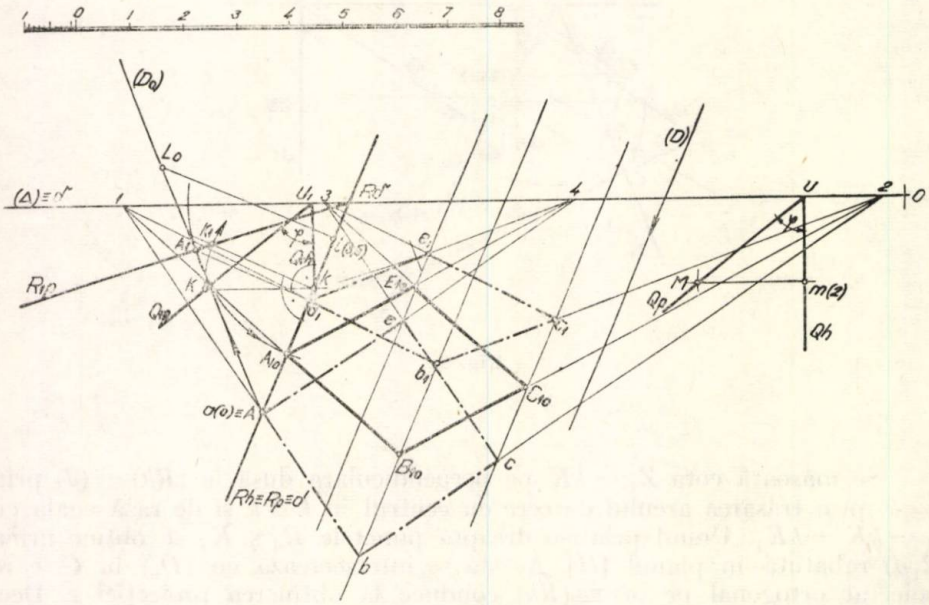

Fig. 3

Se rabate apoi tot în planul reprezentării dreapta proiectantă a vîrfului A. Mai întîi prin (D) se consideră planul vertical $[R]$. Prin urmare $(Rh) \equiv (Rp) \equiv (d)$. Proiectanta (D) se rabate, rabătîndu-i pe planul $[H]$ punctele ce o determină şi care au fost indicate.

Punctul A al dreptei fiind de cotă zero se poate considera $A = (D) \cap [H]$; prin aceasta el este propriul său rabătut, axa de rabatere fiind urma $(Rh) \equiv (d)$. Rabaterea celuilalt punct L, constă în a se duce din $l\,(4,\,5)$ perpendiculara la $(Rh) \neq d$ pe care se măsoară $Z_L = 3,\,5$ unităţi astfel încît $Z_L = lL_0$. Unind printr-o dreaptă punctele L_0 şi A se obţine dreapta (D_0), care este dreapta (D), rabătută în planul reprezentării $[H]$.

Tot planul $[R]$ se intersectează cu planul $[P]$ după dreapta (Rp) care este urma acestuia pe planul de proiecţie şi care se rabate pe planul $[H]$. Punctul $K \in (Rp)$ s-a considerat pe aceeaşi orizontală a planului $[P]$ ca şi punctul $M\,(m\,;\,2)$. Prin acesta s-a construit planu

$[Q_1]$ paralel cu $[Q]$ prin urmare, $k \in (Q_1h) \parallel (Qh)$ şi $K \in (Q_1p) \parallel (Qp)$. Apoi prin proiecţia k se duce perpendiculara pe $(d) \equiv (Rh)$ pe care se măsoară cota punctului K astfel că $kK = kK_1 = Z_K$. Urma rabătută (R_{1p}) în planul $[H]$ este determinată de punctele $R_\delta = (Rh) \cap (\Delta)$ şi de K_1 a cărui construcţie a fost indicată. Cele două drepte coplanare (D_0) şi (R_{1p}) din rabaterea pe planul reprezentării se intersectează în punctul A, proiecţia oblică a vîrfului A pe planul $[P]$. Ridicarea acestui punct în epură înseamnă aducerea dreptei pe care se află în poziţia iniţială (ridicarea ei), ceea ce în epură se reduce la a trasa perpendiculara din A_1 la proiecţia $(d) = (Rh)$ al cărei picior este proiecţia a_1.

Tot punctul A_1 poate fi rotit astfel încît cercul de rotaţie să aibă centrul în punctul R_δ şi să fie situat în planul vertical $[R]$ spre a ajunge în planul $[H]$, pe urma (Rh). De aceea, în rabatere se duce arcul de cerc de centru indicat şi de rază $R_\delta A_1$, ce se intersectează cu $(d) \equiv (Rh)$ în A_{10}, care este una din cele două soluţii ce se pot obţine prin rotaţia indicată.

Considerînd corespondenţa de omologie afină dintre pătratul $abce$ şi proiecţia oblică a acestuia pe planul $[P]$ se trece la construcţia paralelogramelor $a_1b_1c_1e_1$ şi $A_{10}B_{10}C_{10}E_{10}$, care sînt figuri afine ale proiecţiei pătratului dat. Astfel laturile pătratului $abce$ prelungite pînă la axa (Δ), o intersectează în punctele *1*, *2*, *3* şi *4*. Spre exemplu, unind punctul *1* prin drepte cu a_1 şi respectiv cu A_{10}, acestea intersectează proiectanta vîrfului $b \equiv B$ în punctele b_1 şi B_{10}, ca afine ale vîrfului menţionat. Dreptele afine $(b_1\ 2)$ şi $(B_{10}\ 2)$ intersectează proiectanta vîrfului $c \equiv C$ în punctele c_1 şi C_{10}, afinele vîrfului C şi aşa mai departe.

3.2. Proiecţia afină a unui cerc

Fie un cerc situat într-un plan $[\Pi]$ de poziţie generală. Se consideră planul $[\Pi]$ dat prin urmele $(\Pi h])$ şi (Πp) în care se află un cerc Ω de rază cunoscută. Planul $[P]$ de proiecţie, ca şi în alte cazuri, este determinat de axa $(\Delta) \equiv (\delta)$ şi de punctul $M\ (m\ ;\ 3)$. Toate punctele orizontalei planului $[P]$ care trece prin M, vor avea aceeaşi cotă $Z_M = 3$. Se foloseşte această observaţie pentru realizarea rabaterii planului $[\Pi]$ pe planul reprezentării (H). Orizontala planului $[P]$ ce conţine punctul M intersectează urma (Πp) în $L\ (l\ ;\ 3)$, fig. 4. Se construieşte triunghiul de poziţie al acestui punct, care este apoi rabătut în planul reprezentării ocupînd poziţia L_1 (una din soluţiile rabaterii). Dreapta $(\Pi_\delta\ L_1) \equiv (\Pi_p)$ este urma planului $[\Pi]$ din planul $[P]$ rabătută în planul reprezentării $[H]$ şi deci planul $[\Pi]$ este adus prin rabatere să se confunde cu $[H]$.

Se reprezintă cercul Ω în rabatere căruia i se consideră diametrii AB orizontal şi CE măsurat pe linia de cea mai mare pantă a planului $[\Pi]$ în raport cu planul reprezentări ce conţine centrul Ω, deci diametrii perpendiculari sau principali în acest cerc.

Cu ajutorul orizontalei menţionate se face ridicarea centrului, a cărui proiecţie este punctul ω. Pe aceeaşi orizontală ridicată se găseşte axa mare a elipsei $ab = AB$. Proiecţia ce a diametrului CE din rabatere se găseşte cu ajutorul corespondentei afine dintre cercul Ω şi elipsa ω, ca proiecţie paralelă a acestuia pe planul reprezentării $[H]$. Astfel, dreapta BE intersectează urma (Π_h) în α, iar afina ei αb intersectează linia de cea mai mare pantă proiecţie în punctul e. Cum $\omega e = \omega c$ ca semiaxe mici în elipsa ω, rezultă că acesta se poate trasa pe baza axelor găsite. Diagonala ΩI a pătratului circumscris cercului Ω îl intersectează în punctul I, ce are ca corespondente afine proiecţiile i, ca intersecţii dintre diagonalele afine şi dreptele duse, în acest caz, perpendiculare pe axa (Π_h). Odată elipsa ω construită, se proiectează cilindric după o direcţie $D\ (d\ ;\ D_0)$ pe planul de proiecţie P.

În acest scop se duce prin ω dreapta d, care este proiecţia pe planul H, a direcţiei de proiectare. Această dreaptă este determinată de punctele $\Omega(\omega\ ;\ 3)$ şi $N\ (n\ ;\ 2{,}5)$. Deci $d \equiv (\omega n)$. Se consideră prin proiectanta $(\Omega N) \equiv (D)$ planul vertical $[R]$ ajutător, care se intersectează cu planul $[H]$ după dreapta $(Rh) \equiv d$ şi cu planul $[P]$ de proiecţie după urma $(Rp) \equiv (Rh) \equiv (d)$. Urma (Rp) este rabătută pe planul reprezentării $[H]$ în poziţia (R_1p). Pentru aceasta se consideră punctul $K\ (k\ ;\ 3)$ din planul $[P]$, dar situat pe urma $(Rp) \equiv (Rh)$. Se duce perpendiculara din $k(3)$ şi pe această urmă se măsoară $Z_K = K_1k = 3$ unităţi.

Se unesc printr-o dreaptă R_δ şi K_1 care este urma $(R_1\,p)$ rabătută în planul [H]. Se duce prin ω şi prin punctul N rabătut în planul de nivel al centrului Ω, care se intersectează în Ω_1, cu urma R_{1p}. Deci $\Omega_1 = (D_0) \cap (R_{1p})$.

Ducînd din Ω_1 perpendiculara pe planul reprezentării $[H]$, deci în epură pe urma $(Rh) \equiv (Rp) \equiv (d)$ se obţine proiecţia ω_1 pe acest plan a centrului elipsei din planul $[\Pi]$. Dacă acelaşi punct Ω_1 se rabate în planul $[H]$ printr-o rotaţie a acestuia în jurul punctului R_{δ}, astfel încît traiectoria (cercul) rabaterii să rămînă în planul vertical $[R]$, soluţiile găsindu-se în locul în care această traiectorie intersectează urma (Rh), anume în Ω_{10} şi Ω_{20}. În epură s-a adoptat soluţia Ω_{10}.

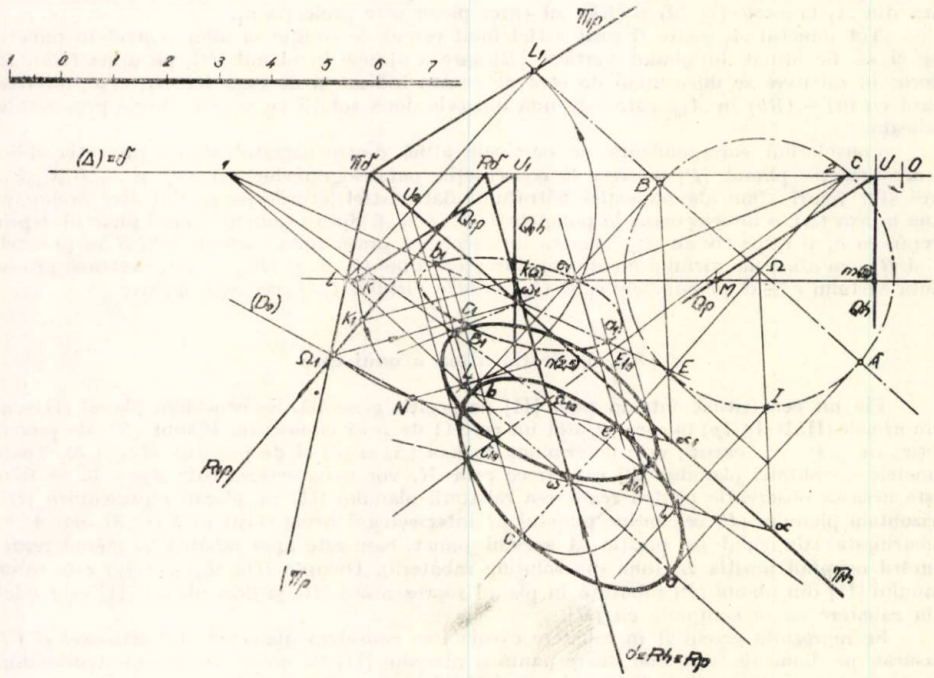

Fig. 4

Considerînd axele elipsei ω ca drepte afine, ele se prelungesc pînă la axa Δ, în punctele 1 şi 2, care unite prin drepte cu centrele afine ω_1 şi Ω_{10} determină dreptele afine ale axelor elipsei ω. Ducînd proiectantele prin extremităţile a, b, c şi e ale axelor, paralele cu direcţia $D\,(d\,;\,D_0)$, se obţin la intersecţia cu dreptele $1\,\omega_1$ şi $2\,\omega_1$; $1\,\Omega_1$ şi $2\,\Omega_{10}$ punctele a_1, b_1, c_1 şi e_1 şi respectiv A_{10}, B_{10}, C_{10} şi E_{10}, care sînt extremităţile diametrilor conjugaţi în elipsele ω_1 şi Ω_{10}, ca segmente afine ale axelor elipsei ω. Cele două elipse ca afine ale cercului Ω se pot astfel trasa.

Observaţii. În construcţia elipselor afine ω_1 şi Ω_{10} se pot găsi şi afinele punctelor în care diagonalele pătratului circumscris cercului Ω intersectează acest cerc; este vorba de punctele I, obţinute la fel cum s-a procedat şi pentru elipsa ω, ca proiecţie pe planul reprezentării $[H]$, a cercului Ω din planul $[\Pi]$.

4. Concluzii

1. Proiecţia paralelă oblică utilizată ca metodă în construcţia imaginilor (proiecţiilor) tridirecţionale ale obiectelor din spaţiu au ca sistem de referinţă diedrul variabil. Planul reprezentării $[H]$ este considerat orizontal

şi de cotă zero, iar celălalt este planul de proiecţie notat, spre exemplu, cu $[P]$.

2. Construcţia proiecţiei din planul $[P]$ s-a făcut cu ajutorul proiecţiei ortogonale din planul reprezentării a obiectului dat pe baza corespondenţei afine a proiecţiilor.

3. Metoda de lucru indicată pentru construcţia plană a proiecţiilor obiectelor date sau concepute este deosebit de utilă pentru reprezentarea tridirecţională a detaliilor din domeniul construcţiilor, arhitecturii, construcţiilor de maşini, etc.

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KORRESPONDENZ-DARSTELLUNGSGEOMETRIE IN DER SCHIEFEN ZYLINDRISCHEN PROJEKTION

(Zusammenfassung)

In dieser Arbeit wird die Konstruktion der schiefen zylindrischen Projektion ebener geometrischer Elemente und Figuren, mit Anwendung des variablen Dieders als Referenzsystem angegeben.

Da zwischen der senkrechten Projektion an der Darstellungsebene $[H]$ und der schiefen zylindrischen Projektion an der Projektionsebene $[P]$, sich eine affine Korrespondenz bildet, wendet man es beim konstruieren der Projektion desselben Elementes oder der vorausgegebenen geometrischen Figur an der Ebene $[P]$, an.

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CALCUL DE CHAMPS EN PHYSIQUE MATHÉMATIQUE PAR LA MISE EN OEUVRE NUMÉRIQUE DES MÉTHODES DU POTENTIEL POUR DES PROBLÈMES PERMANENTS

PAR

C. ROLLAND et A. HAURAT

0. Introduction

La théorie du potentiel trouve ses applications dans de nombreux problèmes de physique mathématique : électricité, chaleur, élasticité ... Elle présente l'intérêt de substituer à la solution d'un problème aux dérivées partielle la résolution d'une équation intégrale. La méthode du potentiel, comme la méthode des sources, est bien connue des ingénieurs et physiciens, mais son application est limitée à des cas particuliers, où la solution formelle est connue.

La mise en oeuvre de la méthode du potentiel dans des cas assez généraux, est limitée par les difficultés numériques rencontrées dans la résolution des équations intégrales introduites par cette méthode. Ces équations intégrales peuvent être singulières (problème de Neumann intérieur) ou d'étude délicate (équation intégrale double, à la fois de „Volterra" et de „Fredholm") et l'approximation des intégrales intervenant dans ces équations peut elle même présenter des difficultés dues aux singularités des noyaux. Ces raisons expliquent que la théorie du potentiel n'ait pas, auprès des utilisatuers, le développement qu'elle mérite.

L'étude des équations intégrales est moins avancée, sur la plan „analyse numérique" que celle des équations aux dérivées partielles et il a paru utile de faire état des techniques utilisées et des résultats numériques que nous avons obtenus. Le manque de place ne permet de présenter que très peu de résultats. Les lecteurs intéressés peuvent se reporter aux travaux que nous résumons içi.

Le contrôle de la qualité des approximations a été obtenu en appliquant ces méthodes à des problèmes dont nous connaissions la solution stricte. Il reste évidemment, dans une phase ultérieure, à établir la stabilité et la convergence des processus.

Bien que cette théorie soit particulièrement bien adaptée aux problèmes „extérieurs", c'est dans le cas de problèmes intérieurs (où les difficultés numériques sont les mêmes) que nous développerons ces techniques.

1. Problèmes permanents bi-dimensionnels

1.1 Recherche d'une fonction harmonique vérifiant une condition de Neumann

Rappel : on se propose de trouver une fonction $\varphi(M)$, régulière dans D, harmonique et vérifiant sur le contour C de D une condition $\frac{d\varphi}{dn_p} = \psi(P)$ donné.

On sait :

1°) que la solution peut être représentée par l'intégrale „de simple couche"

$$\varphi(M) = \int_c g(Q) \operatorname{Log} | QM | \, dQ, \tag{1}$$

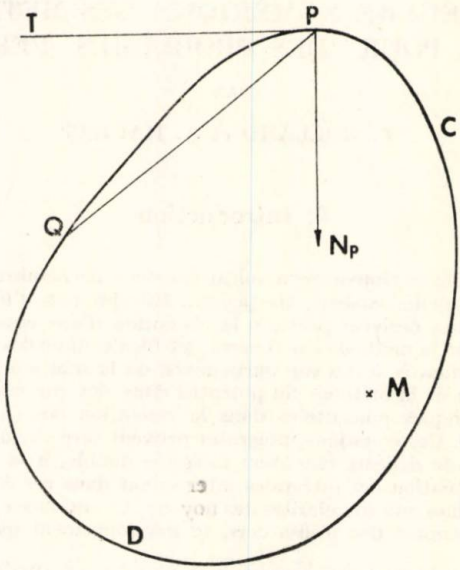

Fig. 1

où $g(P)$ est la „densité" de la simple couche ; cette densité est solution de

$$\pi g(P) + \int_c g(Q) \frac{d}{dn_p} \operatorname{Log} | PQ | \, dQ = \psi(P) \tag{2}$$

aux points P et $Q \in C$.

L'équation (2) n'est soluble que si $\psi(P)$ vérifie la condition

$$\int_c \psi(Q) \, dQ = 0 ; \tag{3}$$

lorsque (3) est vérifiée (2) possède une infinité de solutions, définies à une constante arbitraire près.

Il existe donc une fonctiin $g^*(P)$ telle que

$$\pi g^*(P) + \int_c g^*(Q) \frac{\mathrm{d}}{\mathrm{d}n_p} \operatorname{Log} | PQ | \, \mathrm{d}Q = 0 ; \tag{4}$$

cette densité correspond à la solution évidente $\psi(P) =$ constante, champ harmonique, à dérivée normale nulle sur C.

Donc :

1°) lorsque (3) est vérifiée la solution générale de (2) est $g_1(P) + \lambda g^*(P)$, où g est une solution particulière et où λ est une constante arbitraire.

2°)

$$\int_c g^*(Q) \operatorname{Log} | QM | \, \mathrm{d}Q = V^*, \tag{5}$$

valeur constante tant que M reste sur D ou sur C.

1.2. Étude numérique

Nous supposons que C est un contour régulier (sans point anguleux), et que nous connaissons une représentation paramétrique définissant x et y, coordonnées de P, en fonctions périodiques (de période 2π) d'un angle θ. Au point courant Q correspond le paramètre t.

Le calcul effectif de $\frac{\mathrm{d}}{\mathrm{d}n_p} \log(PQ)$ en fonction de θ et t ne présente aucune difficulté puisque lorsque $Q \to P$, on sait que [1, annexe I]

$$\lim_{P \to Q} \frac{\mathrm{d}}{\mathrm{d}n_p} \operatorname{Log} | PQ | = -\frac{1}{2R_p},$$

où R_p est le rayon de courbure de C en P. De même nous ferons le calcul de $\mathrm{d}Q = \frac{\mathrm{d}s}{\mathrm{d}\theta} \mathrm{d}t$, S étant l'arc de C. $g(P)$, fonction inconnue deviendra $g_1(\theta)$.

Lorsque nous aurons explicité $\frac{\mathrm{d}}{\mathrm{d}n_p} \operatorname{Log} PQ$ et $\mathrm{d}Q$ en fonction θ, t et $\mathrm{d}t$ l'équation intégrale (2) deviendra

$$\pi g_1(\theta) + \int_0^2 g_1(t) K(\theta, t) \, \mathrm{d}t = \psi_1(t), \tag{6}$$

où $K(\theta, t)$ dépendra du contour C, et sera fonction périodique en θ et en t en période 2π.

Compte tenu de cette périodicité nous approcherons alors [2, p 53]

$$\int_0^{\pi} g_1(t)K(\theta,\ t)\ \mathrm{d}t \quad \text{par} \quad \frac{2\pi}{n+1}\sum_0^n g_1(t_j)\,K(\theta,\ t_j),$$

où

$$t_j = j\,\frac{2\pi}{n+1}\ , \qquad (j = 0,\ 1\ ,\ldots,\ n).$$

Écrivons que l'équation (2) où l'intégrale a été ainsi approchée, est vérifiée pour les $n+1$ valeurs

$$\theta_l = l\,\frac{2\pi}{n+1}\ , \qquad (l = 0,\ 1\ ,\ldots,\ n),$$

nous obtenons les $n+1$ équations linéaires

$$\pi g_1(\theta_l) + \frac{2\pi}{n+1}\sum_0^n g_1(t_j)\,K(\theta_l,\ t_j) = \psi(\theta_l),$$

système que nous noterons

$$A\,.\,g = \psi, \tag{7}$$

où A est la matrice carrée de rang $n+1$, éléments

$$\pi\delta_{jl} + \frac{2\pi}{n+1}\,K(\theta_l,\ t_j), \tag{8}$$

δ_{jl} étant le symbole de Kronecker (j et l variant de 0 à n) g et ψ sont les matrices colonnes dont les éléments sont respectivement les valeurs $g(\theta_l)$ et $\psi(\theta_l)$.

Il est naturel de poser que (7) *ne sera considéré comme approximation acceptable de* (2) *que si ces deux systèmes ont des structures analogues* ; d'une façon plus précise nous n'accepterons (7) comme approximation de (2) que si

1°) (7) n'est soluble qu'à condition que le second membre vérifie une condition qui approche (3);

2°) le système (7) homogène (pour lequel $\psi = 0$) admette une solution non nulle, qui sera alors une approximation de $g^*(0)$.

On peut alors interpréter ceci de deux façons différentes.

1. La propriété classique

$$\int_c \frac{\mathrm{d}}{\mathrm{d}n_Q}\,\mathrm{Log}\,|\,PQ\,|\,\mathrm{d}Q = -\pi$$

(discrétisée comme nous venons de le décrire) entraîne que la somme des éléments de chaque colonne de A doit être pratiquement nulle, ceci étant

d'autant mieux vérifié que l'approximation est meilleure. Ce fait a été vérifié numériquement [2, p. 19].

Dans une première méthode nous introduirons les éléments diagonaux A_i de façon que la somme des éléments de chaque colonne soit nulle. La matrice ainsi construite vérifie strictement les conditions que nous avons imposées, et nous pouvons considérer (2) où A est ainsi défini comme une approche satisfaisante de (7), la solution générale de (2) étant une approximation de la solution générale de (7), et le vecteur propre, solution non nulle de $Ag = 0$ étant une approche satisfaisante de $g^*(\theta)$.

Il faut noter que la comparaison entre les valeurs de A_{ii} calculées directement ou définies par le dernier procédé donne une idée de la qualité de l'approximation.

2. Nous remarquerons ensuite que pour calculer effectivement la solution générale de (2), nous calculerons successivement une solution particulière de $Ag = \psi$, puis une solution particulière de $Ag = 0$. Pour calculer chacune de ces solutions nous pouvons

1°) fixer arbitrairement une valeur $g_1(0)$ par exemple;

2°) supprimer la 1-ère ligne de A et résoudre le système restant de n équations à n inconnues, qui peut s'écrire

$$A_1 g' = \psi - V g_1(0), \tag{9}$$

où V est formé de n éléments de la 1-ère colonne de A (après suppression de la première ligne) et A_1 formé des n dernières colonnes de A, après suppression de la 1-ère ligne.

Nous remarquerons alors que (9) est un système „franc" [1]) et que nous n'introduisons pas grande erreur en laissant aux éléments $A_{11}, A_{22}, \ldots, A_{nn}$ les valeurs initiales.

On est ainsi conduit à la philosophie suivante : compte tenu des propriétés connues de (2) nous cherchons une solution particulière prenant en $\theta = 0$ la valeur $g_1(0)$ donnée. Pour déterminer les n inconnues $g(\theta_l)$ $(l = 1, 2, \ldots, n)$ restantes nous écrirons n fois la condition :

$$\pi g_1(\theta_l) + \frac{2\pi}{n+1} \sum_{j=0}^{n} g_1(t_j) K(\theta_l, t_j) = \psi(\theta_l) \tag{10}$$

pour $l = 1, 2, \ldots, n$.

Ce système de n équations à n inconnues ($g_1(\theta_1)$ à $g_1(\theta_n)$ où $g_1(\theta_n)$ est une donnée, est un système dont la résolution numérique ne présente plus aucune difficulté, et qui permet le calcul d'une solution particulière (lorsque $\psi \neq 0$) et de la solution propre (approche de $g_1^*(\theta)$) lorsque nous faisons $\psi = 0$.

1.3. Autres méthodes de calcul

Nous venons d'exposer en détail une technique utilisée lorsque la fonction inconnue est introduite sous forme d'une table de ses valeurs numériques $g_1(\theta_l)$.

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[1]) c'est-à-dire dont la recherche numérique de la solution est facile et telle que, de plus, la solution varie peu pour de légères variations des éléments de la matrice.

On peut également chercher une approche de la solution sous forme de polynôme trigonométrique en posant :

$$g_1(\theta) = a_0 + a_1 \cos\theta + b_1 \sin\theta + \ldots + a_n \cos n\theta + b_n \sin n\theta.$$

Le calcul ne présente aucune difficulté : compte tenu des propriétés particulières de l'équation intégrale, nous nous donnerons à priori la valeur de a_0 et nous ne prendrons comme inconnues que les $2n$ valeurs a_i et b_i $(i = 1, 2, \ldots, n)$.

1.3.1 L'intégrale $\int_c g(Q) \frac{d}{dn_p} \operatorname{Log} | PQ | \, dQ$ sera approchée par

$$a_0 \int \frac{d}{dn_p} \operatorname{Log} | PQ | \, dQ + \sum_1^n \begin{pmatrix} a_i \\ b_i \end{pmatrix} \int_c \begin{pmatrix} \cos i\theta \\ \sin i\theta \end{pmatrix} \frac{d}{dn_p} \operatorname{Log} | PQ | \, dQ$$

et l'équation intégrale deviendra

$$a_0 + \Sigma\, a_i \cos i\theta + \Sigma\, b_i \sin i\theta +$$

$$(11) \qquad + a_0 \int_c \frac{d}{dn_p} \operatorname{Log} | PQ | \, dQ + \sum_1^n a_i \int_c \cos i\theta \frac{d}{dn_p} \operatorname{Log} | PQ | \, dQ +$$

$$+ \sum_1^n b_i \int_c \sin i\theta \frac{d}{dn_p} \operatorname{Log} | PQ | \, dQ = \psi_1(\theta).$$

Chaque intégrale se calcule sans difficulté par la technique indiquée au premier paragraphe.

On approche donc l'équation intégrale par une expression :

$$(12) \qquad \sum_1^n a_i G_i(\theta) + \sum_1^n b_i H_i(\theta) = \psi_1(\theta) - a_0 K_i(\theta),$$

où le calcul explicite des $3n$ fonctions $G_i(\theta)$, $H_i(\theta)$ et $K_i(\theta)$ ne dépend que du contour C.

Pour déterminer les a_i et b_i $(i = 1, 2, \ldots, n)$ on peut utiliser la méthode de collocation (écrire que la condition (12) est vérifiée en $2n$ valeurs $\theta_1, \theta_2, \ldots, \theta_{2n}$) ce qui conduit à un système linéaire franc, mais l'erreur systématique est alors sensible au choix des valeurs θ_l. Nous avons eu des résultats de meilleure qualité en écrivant que la condition (12) est vérifiée en p valeurs θ_l (où p, supérieur à $2n$, est choisi de l'ordre de $4n$ à $6n$). On obtient alors un système surdéterminé dont la pseudo-solution, obtenue par triangularisation [2, p. 125] définit les a_n et b_n.

Cette méthode a été appliquée à la détermination d'une fonction d'Airy, et le lecteur trouvera les résultats numériques en [1, pp. 91—96].

1.3.2 *Calcul des champs.* Nous venons de voir comment calculer la densité dans le cas le plus délicat. Il faut ensuite calculer le champ en différents points.

Dans les différentes applications que nous avons traitées nous avons constaté que le calcul du champ, facile „loin" du contour, devenait difficile au voisinage de celui-ci, et nous avons été amenés à développer diverses techniques.

Champ de simple couche à l'intérieur et sur le contour. On se propose de calculer

$$(13) \qquad I = \int_c g(Q) \operatorname{Log} | MQ | \, dQ$$

lorsque M est un point intérieur à C ou un point du contour C.

La technique utilisée part de l'existence de la solution caractéristique $g^*(P)$, introduite en **1.1**, vérifiant (4) et telle que

$$\int_C g^*(P) \operatorname{Log} | MQ | \, dQ = V^* = \text{constante} \quad M \in \begin{cases} D \\ C \end{cases};$$

$g^*(P)$ n'est défini qu'à une constante de multiplication près. Nous savons en calculer une approximation satisfaisante (cf paragraphe **1.2**). La valeur de la constante V^* correspond à la solution particulière $g^*(P)$ que nous avons choisie et se calcule sans difficulté en calculant l'intégrale (13) en un point M assez „éloigné" du contour (mais intérieur à ce dernier).

On modifie I de la façon suivante :

$$I = \int_C [g(Q) - Kg^*(P)] \operatorname{Log} | MQ | \, dQ + K \int_C g^*(P) \operatorname{Log} | MQ | \, dQ,$$

soit

$$I = \int_C [g(Q) - Kg^*)P)] \operatorname{Log} | MQ | \, dQ + KV^*. \tag{14}$$

On détermine K pour que lorsque $Q \to P$, $g(Q) - Kg^*(P) \to 0$, donc

$$K = \frac{g(P)}{g^*(P)}.$$

La quantité sous le signe somme est nulle pour $Q = P$ et continue au voisinage de ce point et le calcul numérique de (14) se fait sans difficulté.

Champ de simple couche à l'extérieur du contour. Il s'agit de calculer

$$I_1 = \int g(Q) \operatorname{Log} | MQ | \, dQ \tag{15}$$

lorsque M est extérieur à C.

La difficulté apparaît lorsque M est voisin de C car Log $| MQ |$ est alors une fonction difficile à approcher dont le calcul n'est valable qu'au prix d'un nombre élevé de points.

Si la solution $g(Q)$ est définie par un polynôme trigonométrique d'approximation, nous pouvons calculer le noyau de l'intégrale en autant de points qu'il est nécessaire et il y aura simplement alourdissement du calcul lorsque M devient proche de C.

Si $g(Q)$ est définie sous forme de table numérique (ce qui est le cas si nous avons calculé $g_1(\theta_0)$, $g_1(\theta_1)$,..., $g_1(\theta_1)$ paragraphe **1.1**) ; le nombre de points ou $g(Q)$ est défini est fixe et ne peut pas être trop élevé, pour ne pas alourdir inutilement le calcul de $g(Q)$.

La technique qui a été utilisée récemment avec succés consiste à interpoler la fonction $g(Q)$ définie par la table par un polynôme trigonométrique

$$g(Q) = a_0 + \sum_1^n a_i \cos i\theta + \sum_1^n b_i \sin i\theta$$

et à calculer numériquement (15) en utilisant, alors le nombre de points nécessaire.

Notons que dans le cas d'un champ de double couche, les difficultés rencontées sont les mêmes que précédemment et les techniques employées peuvent être les mêmes.

2. Problèmes traités et conclusions

La théorie du potentiel a été développée numériquement d'une part pour la recherche de fonctions biharmoniques intervenant en élasticité plane (fonction d'Airy) ; d'autre part pour des problèmes de barrières, pour des problèmes harmoniques plans ou de révolution. La fig. 2 représente selon le cas le plan xz ou une section méridienne et on se propose de trouver Φ_1 et Φ_2, harmoniques respectivement dans D_1 et D_2, Φ_1 est donné sur AA', Φ_2 est donné sur C_2. De plus Φ_1 et Φ_2 sont liés sur C_1 par

$$\varepsilon_1 \frac{d\Phi_1}{dn} = \varepsilon_2 \frac{d\Phi_2}{dn}.$$

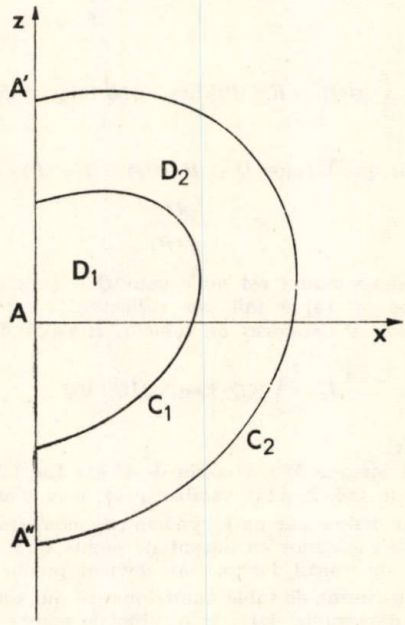

Fig. 2

Pour ne pas alourdir cet article, nous ne citerons que les résultats suivants.

1°) *Problème de Neumann.* Soit l'ellipse d'équations paramétriques $x = a \cos \eta$, $y = b \sin \eta$. On sait que $V(M) = \cos \eta$ est une fonction harmonique, régulière dans l'ellipse de demi-axes a et b, définie par $V(M) = \int_E g(Q) \operatorname{Log} | QM | dQ$ avec $g(Q) = -3x/(4\pi \sqrt{16 - 3x^2})$.

On peut calculer $dV/dn = \alpha/a$ sur l'ellipse (α = cosinus directeur de la normale à l'ellipse) et construire par les techniques exposées une solution V^* harmonique, définie par cette dérivée normale sur l'ellipse. E. On

sait que V^* dépend de la solution $g(Q)$ particulière choisie et que $V^*(M) - V(M)$ doit être constant pour M intérieur à l'ellipse. Le comportement de $V(M) - V^*(M)$ permet donc de caractériser la qualité de l'approximation.

La fig. 3 définit un certain nombre de points de l'ellipse où nous avons calculé $V(M)$ et $V^*(M)$; la solution particulière est caractérisée par le fait que $g(Q) = 0$ au point numéroté „0".

La méthode utilisée est une discrétisation de $g(P)$ en 25 points de l'ellipse correspondants à des valuers de η régulièrement espacées. Les valeurs de $V(M)$, $V^*(M)$ et $V^*(M) - V(M)$ sont reportées dans le tableau 1.

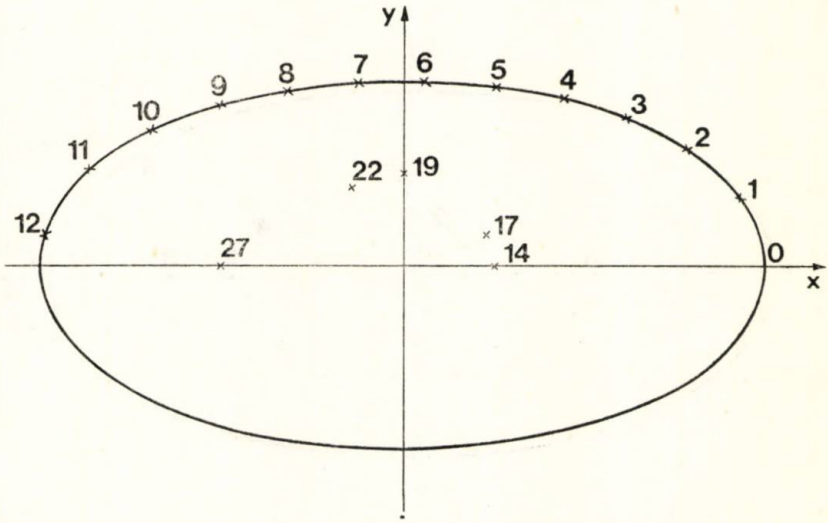

Fig. 3

Tableau 1

N° du point	$V(M)$ Valeur stricte	$V^*(M)$ Valeur calculée	$V^*(M) - V(M)$
1	0,88954847	1,4979262	0,6083778
3	0,46998682	1,0785988	0,6086120
6	0,03154877	0,64022661	0,608677
8	— 0,22873026	0,38025849	0,6089887
12	— 0,96727648	— 0,36098919	0,6072872
14	0,25000000	0,85815602	0,6081560
17	0,21939564	0,82755180	0,6081561
19	0,00019908	0,60835752	0,6081584
22	— 0,10403671	0,50412292	0,0681596
25	— 0,39599700	0,21216691	0,6081639
27	— 0,50000000	1,0816309	0,6081630

2°) *Détermination d'une fonction d'Airy.* Nous étudions la même ellipse que précédemment et partons d'une fonction bi-harmonique

$$\varphi = (p^2 + p^4) \cos 2\eta.$$

On peut calculer φ et $d\varphi/dn$ sur l'ellipse, puis chercher la fonction bi-harmonique φ^* vérifiant ces conditions.

La méthode utilisée consiste à approcher les densités $g_1(\theta)$ et $g_2(\theta)$ par des polynômes trigonométriques jusqu'à l'ordre 16 inclus (termes en cos 16θ et sin 16θ) et à écrire que l'équation intégrale approchée est vérifiée en 112 points (méthode de collocation généralisée ; 112 équations à 34 inconnues, compte-tenu de la symétrie de la solution).

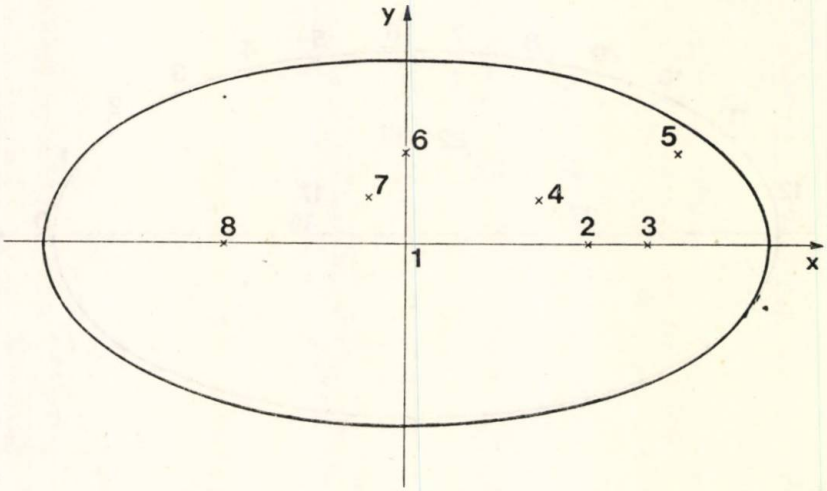

Fig. 4

La fig. 4 définit par leur numéro 8 points intérieurs à l'ellipse. Les résultats sont portés au tableau 2.

Tableau 2

N° du point	Valeur exacte	Valeur calculée	Erreur absolue
1	0	0,00009	10^{-4}
2	0,3125	0,3148	$2\ 10^{-3}$
3	0,4896	0,4924	$3\ 10^{-3}$
4	0,1002	0,1009	$7\ 10^{-4}$
5	0,3944	0,3915	$3\ 10^{-3}$
6	— 0,0664	— 0,0663	10^{-4}
7	— 0,0066	— 0,0065	10^{-3}
8	0,3124	0,3148	$2\ 10^{-5}$

Il faut noter que les équations intégrales introduites dans les problèmes permanents sont des intégrales à noyau régulier, et que la mise en oeuvre de techniques numériques ne rencontrait que des obstacles mineurs.

Le passage aux problèmes évolutifs introduit d'abord une variable supplémentaire, le temps (nous traitons donc, en fait, des équations aux dérivées partielles à 3 dimensions) ; de plus l'équation intégrale fondamentale (qui joue un rôle de (2)) et les intégrales permettant le calcul du champ [rôle analogue à (1)] fait intervenir des noyaux *présentant des singularités mathématiques.*

Nous rencontrerons donc des problèmes nouveaux, liés aux singularités des noyaux des intégrales et à une certaine complexité de l'organisation des programmes de traitement.

Le but de l'article suivant sera de montrer comment de tels problèmes peuvent être résolus.

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CALCULUL CÎMPURILOR ÎN FIZICA MATEMATICĂ PRIN PRELUCRAREA NUMERICĂ A METODELOR POTENŢIALULUI PENTRU PROBLEME PERMANENTE

(Rezumat)

Se expun rezultatele obţinute în studiul pe cale numerică a ecuaţiilor integrale care intervin în problemele de fizică matematică ce utilizează metoda potenţialului.

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RELATIVISTIC MAGNETOHYDRODYNAMICS WITH PLANE SYMMETRY

BY

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1. Introduction

The main equations of relativistic magnetohydrodynamics are the Einstein equations [1]

$$R_{\alpha\beta} - \frac{1}{2} g_\alpha R = k T_{\alpha\beta}, \qquad k = \frac{8\pi G}{c^2}, \tag{1}$$

$$T_{\alpha\beta} = \left(r + \frac{r\varepsilon}{c^2} + \frac{p}{c^2} + \mu\,|h|^2\right) u_\alpha u_\beta - \left(\frac{p}{c^2} + \frac{1}{2}\mu\,|h|^2\right) g_{\alpha\beta} - \mu h_\alpha h_\beta, \tag{2}$$

and the Maxwell equations

$$(u^\alpha h^\beta - u^\beta h^\alpha)_{;\alpha} = 0. \tag{3}$$

We assume that the motion of the fluid is locally adiabatic, $u^\alpha S_{,\alpha} = 0$, or equivalently, that the matter density is conservative, $(ru^\alpha)_{;\alpha} = 0$. The symbols used here have their usual significance [1].

We consider the case when both the space-time determined by the metric tensor $g_{\alpha\beta}$ and the magnetohydrodynamic field determined by the energy tensor $T_{\alpha\beta}$ exhibit plane symmetry [2]. A quantity Q is called plane symmetric (p. s) when it is form-invariant under any infinitesimal transformation given by $\bar{x}^\sigma = x^\sigma + \xi(x^\alpha)$, or in other words, Q is p. s. when it satisfies $\underset{\xi}{\mathcal{L}} Q = 0$ for $\xi = \xi^\alpha_i$, where $\underset{\xi}{\mathcal{L}}$ denotes the Lie derivative with respect to ξ, and

$$\xi^\alpha_i = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & -x^3 \\ 0 & 1 & x^2 \\ 0 & 0 & 0 \end{bmatrix}, \qquad (i = 1,\ 2,\ 3). \tag{4}$$

2. Maxwell Equations

In this section, we prove that the coordinate system in a space-time with plane symmetry in which the line element is given by

$$ds^2 = e^{2F}\,[(dx^4)^2 - (dx^1)^2] - e^{2G}\,[(dx^2)^2 + (dx^3)^2], \tag{5}$$

is a comoving system for a magnetohydrodynamic field in which the velocity vector and the magnetic field have particularly simple forms. In the case of plane symmetry, from the Maxwell Eqs. (3), we obtain

$$u^{\alpha}_{;\alpha}\, h_1 + u^{\alpha}\, h_{1;\alpha} - u_{1;\alpha} h^{\alpha} - u_1 h^{\alpha}_{;\alpha} = 0, \tag{6}$$

$$u^{\alpha}_{;\alpha} h_4 + u^{\alpha} h_{4;\alpha} - u_{4;\alpha} h^{\alpha} - u_4 h^{\alpha}_{;\alpha} = 0, \tag{7}$$

where u_α and h_α satisfy the relations

$$(u_4)^2 - (u_1)^2 = e^{2F} \qquad \text{(unitary character of } u_\alpha), \tag{8}$$

$$u_4 h_4 - u_1 h_1 = 0 \qquad (h_\alpha \text{ is orthogonal to } u_\alpha). \tag{9}$$

We have

$$\begin{aligned} u^{\alpha}_{;\alpha} &= 2[(F_{,1} + G_{,1})u^1 + (F_{,4} + G_{,4})u^4] + u^1_{,1} + u^4_{,4}\,, \\ h^{\alpha}_{;\alpha} &= 2[(F_{,1} + G_{,1})h^1 + (F_{,4} + G_{,4})h^4] + h^1_{,1} + h^4_{,4}\,, \end{aligned} \tag{10}$$

and substituting these into (6) and (7) we find that

$$F_{,1} - G_{,1} = \frac{1}{2}\,[\ln\,(u_1 h_4 - u_4 h_1)]_{,1}\,, \tag{11}$$

$$F_{,4} - G_{,4} = \frac{1}{2}\,[\ln(u_1 h_4 - u_4 h_1)]_{,4}\,. \tag{12}$$

If then we substitute for h_4 in Eqs. (11) and (12) the expression for this quantity gotten from Eq. (9), we obtain

$$= 2G - \ln\left(\frac{h_1}{u_4}\right) - \ln \mathring{h}, \qquad \mathring{h} = \text{constant}. \tag{13}$$

We take $h_1 = \mathring{h} e^{F-2G}$, $u_4 = e^F$. It then follows that

$$\begin{aligned} &h^1 = -\mathring{h} e^{-F-2G}, \qquad u_1 = u^1 = 0, \qquad h^4 = h_4 = 0, \\ &u^4 = e^{-F}, \qquad |h|^2 = -h_\alpha h^\alpha = -h_1 h^1 = \mathring{h}^2\, e^{-4G} \end{aligned} \tag{14}$$

after having made use of Eqs. (8) and (9). One sees immediately that we can introduce the comoving tetrads of orthonormal vectors $\lambda^{\alpha}_{(n)}$ ($n = 1, 2, 3, 4$, labels the vectors) as follows

(15) $$\lambda^{\alpha}_{(n)} = (-e^{-F}\delta_{1n}, e^{-G}\delta_{2n}, e^{-G}\delta_{3n}, e^{-F}\delta_{4n}),$$

where $\lambda^{\alpha}_{(1)} = \dfrac{h^{\alpha}}{|h|}$, $\lambda^{\alpha}_{(4)} = u^{\alpha}$, and δ is Kronecker symbol.

3. Equations of Motion

We consider now the equations of motion of the fluid, $T^{\alpha}_{\beta;\alpha} = 0$, which are consequences of the Einstein Eqs. (1). Using (2), (10) and (14), we find

(16) $$\frac{(r + r\varepsilon/c^2)_{,4}}{r + r\varepsilon/c^2 + p/c^2} = -F_{,4} - 2G_{,4},$$

(17) $$\frac{p_{,1}}{rc^2 + r\varepsilon + p} = -F_{,1},$$

(18) $$\frac{r_{,4}}{r} = -F_{,4} - 2G_{,4}.$$

If the motion of the fluid is such that $r + r\varepsilon/c^2$ is a function of $r + r\varepsilon/c^2 + p/c^2$ alone, then we can use Eqs. (16)—(18) to find p and $rc^2 + r\varepsilon + p$ as functions of F and G. We shall illustrate the method of dealing with these equations for the special case of the degenerate gas („limiting hot gas" [3]) where $rc^2 + r\varepsilon = 3(rc^2 + r\varepsilon + p)/4$. Eqs. (16) and (17) then become

(19) $$\frac{3p_{,4}}{4p} = -F_{,4} - 2G_{,4}, \qquad \frac{p_{,1}}{4p} = -F_{,1}.$$

We distinguish three cases when these equations may be integrated.

i) *Spatially independent case.* In this case all quantities are independent of the variable x^1 and from Eqs. (18) and (19) we obtain

(20) $$r = \mathring{r}e^{-F-2G}, \qquad p = \mathring{p}e^{-4(F+2G)/3}, \qquad \mathring{p},\ \mathring{r} = \text{const.}$$

We note that, the classically relation holds : $p/\mathring{p} = (\mathring{r}/r)^{\varkappa}$.

ii) *Conformal case.* In this case $F = G$ and hence

(21) $$p = \mathring{p}e^{-4F} \qquad \text{with} \qquad F = F(x^1, x^4).$$

iii) *Static case.* If in Eqs. (17) the dependent variable are independent of x^4, then

(22) $$p = \mathring{p}e^{-4F} \qquad \text{with} \qquad F = F(x^1).$$

4. Field Equations

In view of Eqs. (14), the field equations (1) reduce to

$$
\begin{aligned}
&R_4^4 - R_1^1 = krf, \qquad R_1^1 - R_2^2 = k\mu \mid h \mid^2, \\
(23) \qquad &R_1^1 + R_4^4 = 2k\left(\frac{p}{c^2} + \frac{1}{2} \mid \mu \mid h\right)^2, \qquad R_1^4 = 0,
\end{aligned}
$$

where $rfc^2 = rc^2 + r\varepsilon + p$, and

$$
\begin{aligned}
&R_{11} = F_{,44} - F_{,11} + 2[F_{,1} G_{,1} + F_{,4} G_{,4} - (G_{,1})^2 - G_{,11}], \\
&R_{44} = F_{,11} - F_{,44} + 2[F_{,1} G_{,1} + F_{,4} G_{,4}, - (G_{,4})^2 - G_{,44}], \\
(24) \qquad &R_{22} = R_{33} = e^{2G-2F} [G_{,44} - G_{,11} + 2(G_{,4})^2 - 2(G_{,1})^2], \\
&R_{14} = 2(F_{,1} G_{,4} + G_{,1} F_{,4} - G_{,1} G_{,4} - G_{,14}).
\end{aligned}
$$

The right-hand sides of Eqs. (23) are functions of F and G as determined in Sections **2** and **3**.

We restrict ourselves to the static case in this paper. Using Eqs. (22) and (23), as the functions F and G depend on the coordinate x^1 only, we obtain

$$
(25) \qquad 2F_{,1} G_{,1} - G_{,1}^2 - G_{,11} = \frac{2\mathring{p}ke^{-2F}}{c^2},
$$

$$
(26) \qquad F_{,11} + G_{,}^2 + G_{,11} = \frac{k\mathring{p}e^{-2F}}{c^2} + \frac{1}{2} k\mu\mathring{h}^2 e^{2F-4G},
$$

$$
(27) \qquad F_{,11} + G_{,11} - 2F_{,1} G_{,1} = k\mu\mathring{h}^2 e^{2F-4G}.
$$

The fact that this system of three ordinary differential equations is consistent for the two unknown functions F and G is a consequence of the Bianchi identities and the equations of motion which in our case are identically satisfied by our construction.

The system of differential Eqs. (25)—(27) is equivalent to the folowing system

$$
(28) \qquad A = G_{,1}^2 + 2F_{,1} G_{,1} - \frac{k\mathring{p}e^{-2F}}{c^2} + \frac{1}{2} k\mu\mathring{h}^2 e^{2F-4G} = 0,
$$

$$
(29) \qquad B = G_{,1}^2 + G_{,1} - 2F_{,1} G_{,1} + \frac{2k\mathring{p}e^{-2F}}{c^2} = 0,
$$

$$C = F_{,11} + 2F_{,1}G_{,1} - \frac{3k\mathring{p}e^{-2F}}{c^2} - \frac{1}{2}k\mu\mathring{h}^2e^{2F-2G}. \tag{30}$$

The expressions A, B, C satisfy the differential equation

$$A_{,1} + 2G_{,1}(A - B - C) - 2F_{,1}(A + B) = 0. \tag{31}$$

It then follows that it is sufficiently to solve the two equations $A = 0$, $B = 0$, and determine $F(x^1)$ and $G(x^1)$. The functions F and G thus obtained must satisfy the third equation $C = 0$ as a consequence of (31).

Some solutions and a discussion will be published at a later date.

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MAGNETOHIDRODINAMICĂ RELATIVISTĂ CU SIMETRIE PLANĂ

(Rezumat)

Presupunînd că spaţiul-timp şi cîmpul magnetohidrodinamic posedă simetrie plană sînt obţinute ecuaţiile lui Maxwell, ecuaţiile de mişcare şi ecuaţiile de cîmp în cazul existenţei tetrazilor comobili de vectori ortonormali. Se arată că ecuaţiile de mişcare pot fi integrate dacă fluidul magnetohidrodinamic este un gaz degenerat în sensul dat de Taub.

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A SIMPLE EXAMPLE OF FLOW INDUCED BY ELECTROMAGNETIC FORCES

BY

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1. Introduction

In recent times, there has been some interest in the study of the flow produced solely by electromagnetic forces. This is achieved by a suitable combination of the applied magnetic field and the flow geometry. Moffatt [1] demonstrated, how a magnetic field, which is caused to rotate with a uniform angular velocity can generate a velocity field in a conducting fluid contained in a circular cylinder. Sneyd [2] considered the cases of a suddenly applied magnetic field and an oscillating field, applied transverse to the axis of a liquid filled cylinder. The curvature of the boundary enclosing the offluid gives rise to Lorentz force which causes flow. The of im portance of the study of such problems lies in the fact, that it provides a simple means of simulating in the laboratory, the problems of astrophysical and geophysical interest.

In the present paper we have shown a simple way of creating a flow field on a flat plate by electromagnetic forces. In this way the difficulties arising due to curvature of the boundary are avoided. We consider a semi-infinite mass of an electrically conducting fluid bounded below by a thin magnetised plate whose magnetic field oscillates harmonically in time. In addition, there exists every where an externally applied uniform magnetic field perpendicular to the plate. Because of the finite diffusivity of the fluid, the oscillating field of the plate diffuses into the fluid. This gives rise to the Lorentz force in the direction of the field of the plate which causes flow. It is found that across the magneto-viscous layer, in the immediate vicinity of the plate, the induced magnetic field remains spatially uniform, while the velocity field corresponds to a damped shear wave. The effective

decay length of the waze is $\sqrt{\eta\nu/a^2}$, where $a^2 = \frac{1}{2}\left[A_0^2 + \sqrt{A_0^4 + \omega^2\eta^2}\right]$, A_0 is the Alfven wave velocity, ω and η are respectively the frequency of oscillation of the magnetic field and magnetic diffusivity of the fluid. In the magneto-inviscid layer, both the magnetic and the velocity fields propagate as diffused A l f v e n waves. The resultant magnetic field in the entire fluid region will be rotating with a time dependent angular velocity.

2. Formulation and Solution

Consider the motion of a viscous electricaly conducting fluid occupying the half space $y \geqslant 0$, under the action of a uniform magnetic field H_0 applied in the y-direction. The fluid is bounded by an infinite flat plate placed at $y = 0$. The plate is magnetised such that it has a magnetic field fluctuating harmonically in time. Such a field can be created by joining the ends of the plate at large z to an a. c. generator [3]. The direction of the field so produced will be the directiin of the x-axis on the plate. The current induced in the z-direction will interact with the transverse magnetic field giving rise to Lorentz force in the xy-plane. The y-component of the Lorentz force being irrotational will only modify the pressure while the x-component will cause the flow.

Since the plate is infinite in extent the physical variables will depend only on y and t. The velocity and the magnetic fields in the fluid may reasonably be assumed as $\vec{V}(u, 0, 0)$ and $\vec{H}(h, H_0, 0)$. The current density vector J will have only z-component.

The system of equations governing the flow and the magnetic field are

$$\frac{\partial u}{\partial t} = \nu \frac{\sigma^2 u}{\partial y^2} + \frac{\mu H_0}{4\pi e}\frac{\partial h}{\partial y}, \tag{1}$$

$$\frac{\partial h}{\partial t} = \eta \frac{\partial^2 h}{\partial y^2} + H_0 \frac{\partial u}{\partial y}, \tag{2}$$

$$\frac{\partial}{\partial y}\left[p + \frac{\mu H_0}{8\pi\rho}|\vec{H}|^2\right] = 0, \tag{3}$$

where μ, ν, η are respectively magnetic permeability, kinematic viscosity and magnetic diffusivity of the fluid.

(1) and (2) when solved under the boundary conditions:

$$u = 0 \quad \text{at} \quad y = 0,$$

$$h = H_1 \cos \omega t \quad \text{at} \quad y = 0,$$

where H_1 is a constant, and

$$u \to 0, \quad h \to 0 \quad \text{as} \quad y \to \infty,$$

yield [4],

$$u = \frac{H_1 A_0^2 \, e^{i\omega t}}{H_0(\alpha_1 - \alpha_2)} \left[e^{-my} - e^{-ny} \right], \tag{4}$$

$$h = \frac{H_1 \, e^{i\omega t}}{(\alpha_1 - \alpha_2)} \left[\alpha_1 \, e^{-my} - \alpha_2 \, e^{-ny} \right], \tag{5}$$

and the z-component of the current

$$J_z = -\frac{\partial h}{\partial y} = -\frac{H_1 \, e^{i\omega t}}{(\alpha_1 - \alpha_2)} \left[a_1 m \, e^{-my} - \alpha_2 n e^{-ny} \right], \tag{6}$$

where $A_0^2 = \mu H_0^2 / 4\pi\rho$ is the square of Alfven wave velocity,

$$\alpha_1 = \frac{i\omega}{m} - m\nu, \quad \alpha_2 = \frac{i\omega}{n} - n\nu,$$

and

$$m, \, n = \frac{1}{2\sqrt{\eta\nu}} \left[\sqrt{A_0^2 + i\omega\eta(1 + \sqrt{\nu/\eta})^2} \pm \sqrt{A_0^2 + i\omega\eta(1 - \sqrt{\nu/\eta})^2} \right].$$

3. Viscous Boundary Layer and the Flow Outside

For any fixed $\eta \neq 0$, the boundary layer solution near the wall is found by taking the limit as ν and y tend to zero while $y\sqrt{\nu}$ remains finite. In the limit $\nu \to 0$

$$m\sqrt{\nu} = \sqrt{\frac{A_0^2 + i\omega\eta}{\eta}}, \tag{7}$$

and

$$n = \frac{i\omega}{\sqrt{A_0^2 + i\omega\eta}}. \tag{8}$$

Hence we get

$$u = \frac{H_1 A_0^2 e^{i\omega t}}{H_0 \sqrt{A_0^2 + i\omega\eta}} \left[e^{-y\sqrt{(A_0^2 + i\omega\eta)/\eta\nu}} - 1 \right], \tag{9}$$

$$h = H_1 \, e^{i\omega t}. \tag{10}$$

These results depict an oscillating boundary of thickness $\delta \sim O(\sqrt{\eta\nu/a^2})$ near the wall, where $a + ib = \sqrt{A_0^2 + i\omega\eta}$,

$$a,\ b = \left[\frac{1}{2}\left(\pm A_0^2 + \sqrt{A_0^4 + \omega^2\eta^2}\right)\right]^{1/2}. \tag{11}$$

It is interesting to note that within the boundary layer the magnetic field remains spatially constant.

In the limit $\omega \to 0$, we find from (11), $b \to 0$, $a \to A_0$, which gives

$$u = \frac{H_1 A_0}{H_0}\left(e^{-y/\delta_H} - 1\right), \tag{12}$$

where

$$\delta_H = \sqrt{\frac{4\pi e \eta \nu}{\mu H_0^2}}.$$

This is just the case of classical Hartmann boundary layer with boundary layer thickness of the order of the Hartmann depth δ_H.

For the case $\omega \to \infty$, $a,\ b \to \sqrt{\omega\eta/2}$ and we have

$$u = \frac{H_1 A_0^2 (1 - i)}{H_0\sqrt{2\omega\eta}}\left[e^{i(\omega t - y\sqrt{\omega/2\nu}) - y\sqrt{\omega/2\nu}} - e^{i\omega t}\right], \tag{13}$$

which represents, the classical shear wave flow with penetration depth of the order of $\sqrt{\omega/\nu}$.

Flow outside the boundary layer can be obtained by keeping y and η fixed and letting $\nu \to 0$. This gives

$$u = -\frac{A_0^2 H_1}{H_0\sqrt{A_0^2 + i\omega\eta}}\, e^{i\omega(t - y/\sqrt{A_0^2 + i\omega\eta})}, \tag{14}$$

$$h = H_1\, e^{i\omega(t - y/\sqrt{A_0^2 + i\omega\eta})}. \tag{15}$$

This is the magneto-inviscid layer. The velocity and the magnetic field in this layer propagate as diffused Alfven waves travelling away from the plate with velocity $(a^2 + b^2)/b$. The thickness of this layer will be of the order of $(a^2 + b^2)/(a\omega)$. It can be easily shown that the thickness of the magneto-inviscid layer increases as the strength of the transverse field increases. On the other hand the thickness of the magneto-viscous layer decreases with H_0.

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UN EXEMPLU SIMPLU DE CURGERE PRODUSĂ DE FORŢE ELECTROMAGNETICE

(Rezumat)

Se studiază curgerea produsă de o placă magnetizată al cărui cîmp magnetic variază armonic în timp. Se pune în evidenţă că regiunea ocupată de fluid poate fi descompusă în trei straturi distincte. Un strat magneto-vîscos, în care cîmpul de viteze se propagă ca o undă hidromagnetică amortizată, un strat magnetic în care se propagă ca o undă difuzată Alfven şi regiunea exterioară care nu este afectată de prezenţa cîmpului magnetic.

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SAINT-VENANT PROBLEM FOR BEAMS WITH GENERAL CYLINDRICAL ANISOTROPY

BY

C. I. BORŞ

To the memory of my Father I. Gh. Borş (1898–1973)

1. General Equations and Statement of the Problem

Let us consider a beam limited by a cylindrical surface $\mathcal{F}$ and by two planes $x_3 = 0$, $x_3 = h$ $(h > 0)$. We shall denote by $\mathcal{V}$ the region occupied by the beam and by S the domain of a cross-section. We denote by Γ the boundary of S.

Because the material of the beam is with cylindrical anisotropy it is preferable to use a cylindrical sistem of coordinates.

We suppose that there are no body forces so that the equilibrium equations are given by

$$
(1.1)\qquad \begin{cases}
\dfrac{\partial \sigma_{rr}}{\partial r} + \dfrac{1}{r}\dfrac{\partial \sigma_{r\theta}}{\partial \theta} + \dfrac{\partial \sigma_{r3}}{\partial x_3} + \dfrac{\sigma_{rr} - \sigma_{\theta\theta}}{r} = 0, \\[2ex]
\dfrac{\partial \sigma_{r\theta}}{\partial r} + \dfrac{1}{r}\dfrac{\sigma_{\theta\theta}}{\partial \theta} + \dfrac{\partial \sigma_{\theta 3}}{\partial x_3} + 2\dfrac{\sigma_{r\theta}}{r} = 0, \\[2ex]
\dfrac{\partial \sigma_{r3}}{\partial r} + \dfrac{1}{r}\dfrac{\partial \sigma_{\theta 3}}{\partial \theta} + \dfrac{\partial \sigma_{33}}{\partial x_3} + \dfrac{\sigma_{r3}}{r} = 0, \qquad \text{in } \mathcal{V}.
\end{cases}
$$

The lateral surface is supposed to be free from tractions so that we have the boundary conditions

$$
(1.2)\qquad \begin{cases}
\sigma_{rr}\cos(n,\ r) + \sigma_{r\theta}\cos(n,\ \theta) = 0, \\
\sigma_{r\theta}\cos(n,\ r) + \sigma_{\theta\theta}\cos(n,\ \theta) = 0, \\
\sigma_{r3}\cos(n,\ r) + \sigma_{\theta 3}\cos(n,\ \theta) = 0, \quad \text{on } \mathcal{F},
\end{cases}
$$

where $\cos(n, r)$, $\cos(n, \theta)$ are the direction cosines of the exterior normal to the surface. $\mathcal{F}$:

On the end $x_3 = 0$ one gives the components of the resultant and the components of the resultant moment:

$$\overline{R} = (R_1, R_2, R_3), \quad \overline{\mathfrak{M}} = (\mathfrak{M}_1, \mathfrak{M}_2, \mathfrak{M}_3) \quad \text{on} \quad x_3 = 0. \tag{1.3}$$

On the end $x_3 = h$ there are applied tractions so that the beam is in equilibrium. The components R_i and $\mathfrak{M}_i$ are given in terms of the stress components by the formulae

$$\begin{cases} R_1 = -\iint\limits_S (\sigma_{r3} \cos\theta - \sigma_{\theta 3} \sin\theta)\, d\sigma, \\ R_2 = -\iint\limits_S (\sigma_{r3} \sin\theta + \sigma_{\theta 3} \cos\theta)\, d\sigma, \\ R_3 = -\iint\limits_S \sigma_{33}\, d\sigma, \quad \mathfrak{M}_1 = -\iint\limits_S \sigma_{33} r \sin\theta\, d\sigma, \\ \mathfrak{M}_2 = \iint\limits_S \sigma_{33} r \cos\theta\, d\sigma, \quad \mathfrak{M}_3 = -\iint\limits_S r\sigma_{\theta 3}\, d\sigma, \end{cases} \tag{1.4}$$

the stresses σ_{r3}, $\sigma_{\theta 3}$, σ_{33} being taken for $x_3 = 0$.

The material of the beam is supposed to be homogeneus and with general cylindrical anisotropy. Then there is an axiscalled „anisotropy line" so that in a cylindrical system of coordinates with the line of anisotropy as x_3-axis the relations between the stress components and the strain components are given by

$$\| \sigma_{rr}, \sigma_{\theta\theta}, \sigma_{33}, \sigma_{\theta 3}, \sigma_{r3}, \sigma_{r\theta} \|^* = \begin{Vmatrix} c_{11} & c_{12} & c_{13} & c_{14} & c_{15} & c_{16} \\ c_{12} & c_{22} & c_{23} & c_{24} & c_{25} & c_{26} \\ c_{13} & c_{23} & c_{33} & c_{34} & c_{35} & c_{36} \\ c_{14} & c_{24} & c_{34} & c_{44} & c_{45} & c_{46} \\ c_{15} & c_{25} & c_{35} & c_{45} & c_{55} & c_{56} \\ c_{16} & c_{26} & c_{36} & c_{46} & c_{56} & c_{66} \end{Vmatrix} \cdot \begin{Vmatrix} \gamma_{rr} \\ \gamma_{\theta\theta} \\ \gamma_{33} \\ \gamma_{\theta 3} \\ \gamma_{r3} \\ \gamma_{r\theta} \end{Vmatrix}, \tag{1.5}$$

where c_{ij} are constants which characterize the elastic qualities of the material of the beam.

The inverse relation are taken in the form

$$\| \gamma \| = \| s_{ij} \| \, \| \sigma \|, \tag{1.6}$$

where γ is the vector of the strain components γ_{rr},... and σ is the vector of the stress components σ_{rr},....

The strain components $\gamma_{rr}, \ldots, \gamma_{r\theta}$ are connected with the components of the displacements u_r, u_θ, u_3 by

$$\gamma_{rr} = \frac{\partial u_r}{\partial r}, \quad \gamma_{\theta\theta} = \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{u_r}{r}, \quad \gamma_{33} = \frac{\partial u_3}{\partial x_3},$$

$$\gamma_{\theta 3} = \frac{\partial u_\theta}{\partial x_3} + \frac{1}{r}\frac{\partial u_3}{\partial \theta}, \quad \gamma_{r3} = \frac{\partial u_3}{\partial r} + \frac{\partial u_r}{\partial x_3},$$

$$\gamma_{r\theta} = \frac{1}{r}\frac{\partial u_r}{\partial \theta} + \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r}.$$

2. Auxiliary Plane Problems

The solution of the problem of Saint-Venant is closely connected with some generalised plane problems which we shall define below.

Firstly, let us consider a generalised plane problem with body forces defined by

$$(2.1) \qquad \begin{cases} \dfrac{\partial \tau_{rr}}{\partial r} + \dfrac{1}{r}\dfrac{\partial \tau_{r\theta}}{\partial \theta} + \dfrac{\tau_{rr} - \tau_{\theta\theta}}{r} + F_r = 0, \\[2ex] \dfrac{\partial \tau_{r\theta}}{\partial r} + \dfrac{1}{r}\dfrac{\partial \tau_{\theta\theta}}{\partial \theta} + 2\dfrac{\tau_{r\theta}}{r} + F_\theta = 0, \\[2ex] \dfrac{\partial \tau_{r3}}{\partial r} + \dfrac{1}{r}\dfrac{\partial \tau_{\theta 3}}{\partial \theta} + \dfrac{\tau_{r3}}{r} = 0, \quad \text{in} \quad S \end{cases}$$

and

$$(2.2) \qquad \begin{cases} \tau_{rr}\cos(n, r) + \tau_{r\theta}\cos(n, \theta) = p_r, \\ \tau_{r\theta}\cos(n, r) + \tau_{\theta\theta}\cos(n, \theta) = p_\theta, \\ \tau_{r3}\cos(n, r) + \tau_{\theta 3}\cos(n, \theta) = 0, \quad \text{on} \quad \Gamma. \end{cases}$$

The existence of a solution for this plane problem requires that

$$(2.3) \qquad \begin{cases} \displaystyle\int_\Gamma (p_r\cos\theta - p_\theta\sin\theta)\,\mathrm{d}s + \iint_S (F_r\cos\theta - F_\theta\sin\theta)\,\mathrm{d}\sigma = 0, \\[2ex] \displaystyle\int_\Gamma (p_r\sin\theta + p_\theta\cos\theta)\mathrm{d}s + \iint_S (F_r\sin\theta + F_\theta\cos\theta)\,\mathrm{d}\sigma = 0, \\[2ex] \displaystyle\int_\Gamma r p_\theta\,\mathrm{d}s + \iint_S r F_\theta\,\mathrm{d}\sigma = 0. \end{cases}$$

Now, let $t_{rr}^{(k)}, \ldots$ $(k = 1, 2, 3, 4)$ be the stresses of our plane problems and let a, b, c, τ be arbitrary constants and

$$\tau_{rr} = at_{rr}^{(1)} + bt_{rr}^{(2)} + ct_{rr}^{(3)} + \tau t_{rr}^{(4)}, \ldots.$$

The mentioned plane problems are defined by the equations

$$(2.4)\quad \begin{cases} \dfrac{\partial \tau_{rr}}{\partial r} + \dfrac{1}{r}\dfrac{\partial \tau_{r\theta}}{\partial \theta} + \dfrac{\tau_{rr} - \tau_{\theta\theta}}{r} + h_{rr,r} + \dfrac{1}{r} h_{r\theta,\theta} + \dfrac{1}{r}(h_{rr} - h_{\theta\theta}) = 0, \\[2ex] \dfrac{\partial \tau_{r\theta}}{\partial r} + \dfrac{1}{r}\dfrac{\partial \tau_{\theta\theta}}{\partial \theta} + 2\dfrac{\tau_{r\theta}}{r} + h_{r\theta,r} + \dfrac{1}{r} h_{\theta\theta,\theta} + \dfrac{2}{r} h_{r\theta} = 0, \\[2ex] \dfrac{\partial \tau_{r3}}{\partial r} + \dfrac{1}{r}\dfrac{\partial \tau_{\theta 3}}{\partial \theta} + \dfrac{\tau_{r3}}{r} = 0, \quad \text{in} \quad S, \end{cases}$$

and

$$(2.5)\quad \begin{cases} \tau_{rr}\cos(n, r) + \tau_{r\theta}\cos(n, \theta) = -[h_{rr}\cos(n, r) + h_{r\theta}\cos(n, \theta)], \\ \tau_{r\theta}\cos(n, r) + \tau_{\theta\theta}\cos(n, \theta) = -[h_{r\theta}\cos(n, r) + h_{\theta\theta}\cos(n, \theta)], \\ \tau_{r3}\cos(n, r) + \tau_{\theta 3}\cos(n, \theta) = 0, \quad \text{on } \Gamma, \end{cases}$$

where the function

$$\omega = a\varphi_1 + b\varphi_2 + c\varphi_3 + \tau\varphi$$

is defined by

$$(2.6)\qquad \Delta\omega + h_{r3,r}^{*} + \frac{1}{r} h_{r3}^{*} + \frac{1}{r} h_{\theta 3,\theta}^{*} = 0, \quad \text{in } S,$$

$$(2.7)\qquad \mathcal{D}\omega = -[h_{r3}^{*}\cos(n, r) + h_{\theta 3}^{*}\cos(n, \theta)], \text{ on } \Gamma,$$

and

$$(2.8)\quad \begin{cases} \Delta = c_{55}\left(\dfrac{\partial^2}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial}{\partial r}\right) + \dfrac{2}{r} c_{45}\dfrac{\partial^2}{\partial r \partial \theta} + \dfrac{1}{r^2} c_{44}\dfrac{\partial^2}{\partial \theta^2}, \\[2ex] \mathcal{D} = [c_{55}\cos(n, r) + c_{45}\cos(n, \theta)]\dfrac{\partial}{\partial r} + \\[2ex] \quad + \dfrac{1}{r}[c_{45}\cos(n, r) + c_{44}\cos(n, \theta)]\dfrac{\partial}{\partial \theta}, \\[2ex] \dfrac{\partial \Phi}{\partial r} = \Phi_{,r}, \quad \dfrac{\partial \Phi}{\partial \theta} = \Phi_{,\theta}, \\[2ex] h_{rr} = c_{13}(ar\sin\theta + br\cos\theta + c) + c_{14}\left(\dfrac{1}{r}\omega_{,\theta} + \tau r\right) + c_{15}\omega_{,r}, \end{cases}$$

$$
(2.8)\quad \begin{cases}
h_{\theta\theta} = c_{23}(ar\sin\theta + br\cos\theta + c) + c_{24}\left(\dfrac{1}{r}\,\omega_{,\theta} + \tau r\right) + c_{25}\,\omega_{,r}\,, \\[2ex]
h_{33} = c_{33}(ar\sin\theta + br\cos\theta + c) + c_{34}\left(\dfrac{1}{r}\,\omega_{,\theta} + \tau r\right) + c_{35}\,\omega_{,r}\,, \\[2ex]
h_{\theta 3} = c_{34}(ar\sin\theta + br\cos\theta + c) + c_{44}\left(\dfrac{1}{r}\,\omega_{,\theta} + \tau r\right) + c_{45}\omega_{,r}\,, \\[2ex]
h_{r3} = c_{35}(ar\sin\theta + br\cos\theta + c) + c_{45}\left(\dfrac{1}{r}\,\omega_{,\theta} + \tau r\right) + c_{55}\,\omega_{,r}\,, \\[2ex]
h_{r\theta} = c_{36}(ar\sin\theta + br\cos\theta + c) + c_{46}\left(\dfrac{1}{r}\,\omega_{,\theta} + \tau r\right) + c_{56}\,\omega_{,r}\,;
\end{cases}
$$

h^*_{rr} ,... are obtained from h_{rr} ,... by droping out the terms of ω.

By equating with zero the coefficients of a, b, c, τ in the Eqs. (2.4), (2.5) and (2.6), (2.7) we can easy derive the equations for our four plane problems and for the functions φ_1, φ_2, φ_3, φ. For example, the function φ is defined by

$$
(2.9)\qquad \begin{cases}
\Delta\varphi + 2c_{45} = 0, \quad \text{in} \quad S, \\
\mathcal{D}\varphi = -r[c_{45}\cos(n,\ r) + c_{44}\cos(n,\ \theta)], \quad \text{on} \quad \Gamma.
\end{cases}
$$

We shall call this function „the function of torsion". We can prove that the conditions (2.3) are verified by the problem (2.4), (2.5). Also, the boundary value problem (2.6), (2.7) has solution.

In order to solve the plane strain problem (2.4), (2.5) we may take the stresses in the form

$$
\tau_{rr} = -h_{rr} + \frac{1}{r}\,\Phi_{,r} + \frac{1}{r^2}\,\Phi_{,\theta\theta}\,, \qquad \tau_{\theta\theta} = -h_{\theta\theta} + \Phi_{,rr}\,,
$$

$$
\tau_{r\theta} = -h_{r\theta} - \frac{1}{r}\,\Phi_{,r\theta} + \frac{1}{r^2}\,\Phi_{,\theta}\,,
$$

$$
\tau_{\theta 3} = -h_{\theta 3} - \psi_{,r}\,, \quad \tau_{r3} = -h_{r3} + \frac{1}{r}\,\psi_{,\theta}\,,
$$

$$
\begin{aligned}
\tau_{33} = {} & -h_{33} + \frac{1}{s_{33}}\,(a\,r\sin\theta + br\cos\theta + c) - \\
& - \frac{1}{s_{33}}\left[s_{13}\left(\frac{1}{r}\,\Phi_{,r} + \frac{1}{r^2}\,\Phi_{,\theta\theta}\right) + s_{23}\,\Phi_{,rr} - s_{34}\,\psi_{,r} + \right. \\
& \left. + s_{35}\,\frac{1}{r}\,\Psi_{,\theta} - s_{36}\left(\frac{1}{r}\,\Phi_{,r\theta} - \frac{1}{r^2}\,\Phi_{,\theta}\right)\right]
\end{aligned}
$$

in order to satisfy the equilibrium equations (2.4).

The equations for the unknown functions $\Phi(r, \theta)$, $\Psi(r, \theta)$ can be obtained from the compatibility conditions.

3. Saint-Venant Problem

The problem of Saint-Venant, for particular cylindrical anisotropy, was studied by diverse scientists. We should like to point out the papers [1]...[4].

For the general anisotropy the problems of bending by couples, the torsion and the extension were studied by Lekhnitski [1], [5].

Here, by an analogous method as in [5] we shall solve the entire problem of Saint-Venant starting by a representation of displacements given by

$$
(3.1)\quad \begin{cases}
u_r = \left[\Psi_r^{(1)} - \dfrac{1}{6}(a_1 \sin\theta + b_1 \cos\theta)x_3^2\right] x_3 + \\
\quad + \Psi_r^{(0)} - \dfrac{1}{2}(a_0 \sin\theta + b_0 \cos\theta)x_3^2 + U_r, \\
u_\theta = \left[\Psi_\theta^{(1)} + \dfrac{1}{2}\tau_1 r x_3 - \dfrac{1}{6}(a_1 \cos\theta - b_1 \sin\theta)\, x_3^2\right] x_3 + \\
\quad + \Psi_\theta^{(0)} + \tau_0 r x_3 - \dfrac{1}{2}(a_0 \cos\theta - b_0 \sin\theta)\, x_3^2 + U_\theta, \\
u_3 = \left[\Psi_3^{(1)} + \omega_1 + \dfrac{1}{2}(a_1 r \cos\theta + b_1 r \sin\theta + c_1)x_3\right] x_3 + \\
\quad + \Psi_3^{(0)} + \omega_0 + (a_0 r \cos\theta + b_0 r \sin\theta + c_0)\, x_3 + U_3 + \Omega,
\end{cases}
$$

where

$$\Psi_r^{(\alpha)} = a_\alpha u_r^{(1)} + b_\alpha u_r^{(2)} + c_\alpha u_r^{(3)} + \tau_\alpha u_r^{(4)}, \ \ldots,$$

$$\omega_\alpha = a_\alpha \varphi_1 + b_\alpha \varphi_2 + c_\alpha \varphi_3 + \tau_\alpha \varphi, \qquad (\alpha = 1, 2),$$

the functions $u_r^{(k)}$, $u_\theta^{(k)}$, $u_3^{(k)}$ $(k = 1, 2, 3, 4)$ being the displacements of the auxiliary plane problems defined by (2.4), (2.5) and the functions φ_1, φ_2, φ_3, φ are defined by (2.6), (2.7); $\Omega(r, \theta)$, $U_r(r, \theta)$, $U_\theta(r, \theta)$ and $U_3(r, \theta)$ are unknown functions and a_α, b_α, c_α, τ_α are unknown konstants.

We shall choose the function Ω such that the problem which will be obtained for the determination of the displacements U_r, U_θ, U_3 be a plane strain problem. On that account we shall call Ω „function of correction".

The stress components of the displacements (**3.1**) are given by

$$
(3.2)\quad \begin{cases}
\sigma_{rr} = (\tau_{rr}^{(1)} + h_{rr}^{(1)})\, x_3 + (\tau_{rr}^{(0)} + h_{rr}^{(0)}) + \tau_{rr} + H_{rr}\,, \\
\sigma_{\theta\theta} = (\tau_{\theta\theta}^{(1)} + h_{\theta\theta}^{(1)}) x_3 + (\tau_{\theta\theta}^{(0)} + h_{\theta\theta}^{(0)}) + \tau_{\theta\theta} + H_{\theta\theta}\,, \\
\sigma_{33} = (\tau_{33}^{(1)} + h_{33}^{(1)}) x_3 + (\tau_{33}^{(0)} + h_{33}^{(0)}) + \tau_{33} + H_{33}\,, \\
\sigma_{\theta 3} = (\tau_{\theta 3}^{(1)} + h_{\theta 3}^{(1)})\, x_3 + (\tau_{\theta 3}^{(0)} + h_{\theta 3}^{(0)}) + \tau_{\theta 3} + H_{\theta 3}\,, \\
\sigma_{r3} = (\tau_{r3}^{(1)} + h_{r3}^{(1)})\, x_3 + (\tau_{r3}^{(0)} + h_{r3}^{(0)}) + \tau_{r3} + H_{r3}\,, \\
\sigma_{r\theta} = (\tau_{r\theta}^{(1)} + h_{r\theta}^{(1)}) x_3 + (\tau_{r\theta}^{(0)} + h_{r\theta}^{(0)}) + \tau_{r\theta} + H_{r\theta}\,,
\end{cases}
$$

where $\tau_{rr}^{(\alpha)}, \ldots$ and $\tau_{rr}, \ldots$ are the stresses corresponding to the displacements $\psi_r^{(\alpha)}$, $\psi_\theta^{(\alpha)}$, $\psi_3^{(\alpha)}$ and U_r, U_θ, U_3; $h_{rr}^{(\alpha)}, \ldots$ are obtained from $h_{rr}, \ldots$ by replacing the constants a, b, c, τ with a_α, b_α, c_α, τ_α;

$$
(3.3)\qquad \tau_{rr}^{(\alpha)} = a_\alpha\, t_{rr}^{(1)} + b_\alpha\, t_{rr}^{(2)} + c_\alpha\, t_{rr}^{(3)}, + \tau_\alpha\, t_{rr}^{(4)}, \ldots
$$

and

$$
(3.4)\quad \begin{cases}
H_{rr} = c_{13}(\Psi_3^{(1)} + \omega_1) + c_{14}\left(\frac{1}{r}\Omega_{,\theta} + \Psi_\theta^{(1)}\right) + c_{15}\,(\Omega_{,r} + \Psi_r^{(1)}), \\
H_{\theta\theta} = c_{23}(\Psi_3^{(1)} + \omega_1) + c_{24}\left(\frac{1}{r}\Omega_{,\theta} + \Psi_\theta^{(1)}\right) + c_{25}\,(\Omega_{,r} + \Psi_r^{(1)}), \\
H_{33} = c_{33}(\Psi_3^{(1)} + \omega_1) + c_{34}\left(\frac{1}{r}\Omega_{,\theta} + \Psi_\theta^{(1)}\right) + c_{35}\,(\Omega_{,r} + \Psi_r^{(1)}), \\
H_{\theta 3} = c_{34}(\Psi_3^{(1)} + \omega_1) + c_{44}\left(\frac{1}{r}\Omega_{,\theta} + \Psi_\theta^{(1)}\right) + c_{45}\,(\Omega_{,r} + \Psi_r^{(1)}), \\
H_{r3} = c_{35}(\Psi_3^{(1)} + \omega_1) + c_{45}\left(\frac{1}{r}\Omega_{,\theta} + \Psi_\theta^{(1)}\right) + c_{55}\,(\Omega_{,r} + \Psi_r^{(1)}), \\
H_{r\theta} = c_{36}\,(\Psi_3^{(1)} + \omega_1) + c_{46}\left(\frac{1}{r}\Omega_{,\theta} + \Psi_\theta^{(1)}\right) + c_{56}\,(\Omega_{,r} + \Psi_r^{(1)}).
\end{cases}
$$

Choosing the function Ω so that

$$
(3.5)\qquad \Delta\Omega + \tau_{33}^{(1)} + h_{33}^{(1)} + H_{r3,r}^{*} + \frac{1}{r} H_{\theta 3,\theta}^{*} + \frac{1}{r} H_{r3}^{*} = 0, \text{ in } S
$$

and

$$
(3.6)\qquad \mathcal{D}\Omega = -[H_{r3}^{*} \cos(n,\ r) + H_{\theta 3}^{*} \cos(n,\ \theta)], \quad \text{on } \Gamma,
$$

with H^*_{r3}, $H^*_{\theta 3}$ obtained from H_{r3}, $H_{\theta 3}$ by droping out the terms of Ω, then the stresses (3.2) will satisfy the equilibrium equations (1.1) and the boundary conditions (1.2) if the stresses τ_{rr},... verify the equations

$$
(3.7)\quad \begin{cases}
\dfrac{\partial \tau_{rr}}{\partial r} + \dfrac{1}{r}\dfrac{\partial \tau_{r\theta}}{\partial \theta} + \dfrac{\tau_{rr} - \tau_{\theta\theta}}{r} + \tau^{(1)}_{r3} + h^{(1)}_{r3} + \\
\qquad + H_{rr,r} + \dfrac{1}{r} H_{r\theta,\theta} + \dfrac{1}{r}(H_{rr} - H_{\theta\theta}) = 0, \\
\dfrac{\partial \tau_{r\theta}}{\partial r} + \dfrac{1}{r}\dfrac{\partial \tau_{\theta\theta}}{\partial \theta} + 2\dfrac{\tau_{r\theta}}{r} + \tau^{(1)}_{\theta 3} + h^{(1)}_{\theta 3} + \\
\qquad + H_{r\theta,r} + \dfrac{1}{r} H_{\theta\theta,\theta} + \dfrac{2}{r} H_{r\theta} = 0, \\
\dfrac{\partial r_{r3}}{\partial r} + \dfrac{1}{r}\dfrac{\partial \tau_{\theta 3}}{\partial \theta} + \dfrac{\tau_{r3}}{r} = 0 \quad \text{in } S
\end{cases}
$$

and

$$
(3.8)\quad \begin{cases}
\tau_{rr}\cos(n, r) + \tau_{r\theta}\cos(n, \theta) = -[H_{rr}\cos(n, r) + H_{r\theta}\cos(n, \theta)], \\
\tau_{r\theta}\cos(n, r) + \tau_{\theta\theta}\cos(n, \theta) = -[H_{r\theta}\cos(n, r) + H_{\theta\theta}\cos(n, \theta)], \\
\tau_{r3}\cos(n, r) + \tau_{\theta 3}\cos(n, \theta) = 0 \text{ on } \Gamma.
\end{cases}
$$

Now, let us show that we can calculate the constants a_1, b_1, c_1, τ_1 before to solve the problems (3.5), (3.6) and (3.7), (3.8).

The boundary value problem (3.5), (3.6) is of the Neumann type. The existence of the function Ω requires that

$$\iint_S (\tau^{(1)}_{33} + h^{(1)}_{33})\, d\sigma = 0.$$

The Eqs. (3.7), (3.8) define a generalized plane strain problem and for it the conditions (2.3) must be fulfilled. From these conditions we get

$$\iint_S (\tau^{(1)}_{\theta 3} + h^{(1)}_{\theta 3})\, r d\sigma = 0.$$

After some calculus easy to do these equations become

$$
(3.9)\quad \begin{cases}
\iint_S R^{(1)}_{\theta 3}(a_1, b_1, c_1, \tau_1)\, d\sigma = 0, \\
\iint_S R^{(1)}_{33}(a_1, b_1, c_1, \tau_1)\, d\sigma = 0,
\end{cases}
$$

where

$$R_{\theta 3}^{(\alpha)}(a_\alpha,\ b_\alpha,\ c_\alpha,\ \tau_\alpha) = \left\{ rt_{\theta 3}^{(1)} + \left[c_{34}\left(\frac{1}{r}\varphi_{1,\theta} + r\right) + c_{35}\,\varphi_{1,r}\right] r\sin\theta \right\} a_\alpha +$$

$$+ \left\{ rt_{\theta 3}^{(2)} + \left[c_{34}\left(\frac{1}{r}\varphi_{2,\theta} + r\right) + c_{35}\,\varphi_{2,r}\right] r\cos\theta \right\} b_\alpha +$$

$$+ \left\{ rt_{\theta 3}^{(3)} + \left[c_{34}\left(\frac{1}{r}\varphi_{3,\theta} + r\right) + c_{35}\,\varphi_{3,r}\right]\right\} c_\alpha + \left\{ rt_{\theta 3}^{(4)} + c_{44}\left(\frac{1}{r}\varphi_{,\theta} + r\right) + c_{45}\,\varphi_{,r}\right] r \right\} \tau_\alpha ,$$

$$R_{33}^{(\alpha)}(a_\alpha,\ b_\alpha,\ c_\alpha,\ \tau_\alpha) = \left[t_{33}^{(1)} + c_{33}\, r\sin\theta + c_{34}\,\frac{1}{r}\varphi_{1,\theta} + c_{35}\,\varphi_{1,r}\right] a_\alpha +$$

$$\left[+ t_{33}^{(2)} + c_{33}\, r\cos\theta + c_{34}\,\frac{1}{r}\varphi_{2,\theta} + c_{35}\,\varphi_{2,r}\right] b_\alpha +$$

$$+ \left[t_{33}^{(3)} + c_{33} + c_{34}\,\frac{1}{r}\varphi_{3,\theta} + c_{35}\,\varphi_{3,r}\right] c_\alpha + \left[t_{33}^{(4)} + c_{34}\left(\frac{1}{r}\,\varphi_{,\theta} + r\right) + c_{35}\,\varphi_{,r}\right] \tau_\alpha .$$

From the end conditions (1.**3**) we get

$$(3.10)\qquad \begin{cases} \displaystyle\iint_S R_{33}^{(1)}\,(a_1,\ b_1,\ c_1,\ \tau_1)\, r\cos\theta\, d\sigma = -R_1, \\ \displaystyle\iint_S R_{33}^{(1)}\,(a_1,\ b_1,\ c_1,\ \tau_1)\, r\sin\theta\, d\,\sigma = -R_2 \end{cases}$$

and

$$(3.11)\qquad \begin{cases} \displaystyle\iint_S R_{33}^{(0)}\,(a_0,\ b_0,\ c_0,\ \tau_0)\, d\sigma = -R_3 - \iint_S (\tau_{33} + H_{33})\, d\sigma, \\ \displaystyle\iint_S R_{33}^{(0)}\,(a_0,\ b_0,\ c_0,\ \tau_0)\, r\sin\theta\, d\sigma = -\mathfrak{M}_1 - \iint_S (\tau_{33} + H_{33}) r\sin\theta\, d\sigma, \\ \displaystyle\iint_S R_{33}^{(0)}\,(a_0,\ b_0,\ c_0,\ \tau_0)\, r\cos\theta\, d\sigma = \mathfrak{M}_2 - \iint_S (\tau_{33} + H_{33}) r\cos\theta\, d\sigma, \\ \displaystyle\iint_S R_{\theta 3}^{(0)}\,(a_0,\ b_0,\ c_0,\ \tau_0)\, d\sigma = -\mathfrak{M}_3 - \iint_S (\tau_{\theta 3} + H_{\theta 3})\, r d\sigma. \end{cases}$$

Therefore, for the determination of the constants a_1, b_1, c_1, we have obtained the equations (**3.9**), (**3.10**) and for the constants a_0, b_0, c_0, τ_0 the equations (**3.11**).

These equations are linear and the coeficients depend on the auxiliary plane problems and of the functions φ_i. It is easy to see that both systems have the same determinant. We shall call it the determinant of problem.

4. The Study of the Determinant of Problem

In order to simplify the calculus let us suppose that $a_1 = b_1 = c_1 = \tau_1 = 0$. It results $U_r = U_\theta = U_3 = \Omega = 0$.

This case corresponds to the following end conditions:

$$\overline{R} = (0,\ 0,\ R_3),\quad \overline{\mathfrak{M}} = (\mathfrak{M}_1,\ \mathfrak{M}_2,\ \mathfrak{M}_3) \text{ on } x_3 = 0\ .$$

The stresses σ_{rr},... are of the form

$$\sigma_{rr} = a_0\sigma_{rr}^{(1)} + b_0\sigma_{rr}^{(2)} + c_0\sigma_{rr}^{(3)} + \tau_0\sigma_{rr}^{(4)},\ldots$$

In this way we have separated from our problem four problems. The corrresponding displacements can be obtained from the equations (**3.1**). The stresses $\sigma_{rr}^{(k)}$,... verify the equations (**1.1**) and the boundary conditions (**1.2**).

We can discover the stresses $\sigma_{rr}^{(k)}$,... ($k = 1,\ 2,\ 3,\ 4$) in the formulae (**3.2**). Below we need the following stresses:

$$\sigma_{33}^{(i)} = t_{33}^{(i)} + c_{33}x^{(i)} + c_{34}\left(\frac{1}{r}\varphi_{i,\theta} + \lambda_i r\right) + c_{35}\varphi_{i,r}\,,$$

$$\sigma_{\theta 3}^{(i)} = t_{\theta 3}^{(i)} + c_{34}x^{(i)} + c_{44}\left(\frac{1}{r}\varphi_{i,\theta} + \lambda_i r\right) + c_{45}\varphi_{i,r}\,,$$

$$\sigma_{r3}^{(i)} = t_{r3}^{(i)} + c_{35}x^{(i)} + c_{45}\left(\frac{1}{r}\varphi_{i,\theta} + \lambda_i r\right) + c_{55}\varphi_{i,r}\,,$$

$$(i = 1,\ 2,\ 3,\ 4\,;\ x^{(1)} = r\sin\theta,\ x^{(2)} = r\cos\theta,\quad x^{(3)} = 1,\ x^{(4)} = 0,$$

$$\lambda_1 = \lambda_2 = \lambda_3 = 0,\quad \lambda_4 = 1,\quad \varphi_4 = \varphi).$$

Let Σ be the whole boundary of the beam and let us consider

$$U_{\alpha\beta} = U_{\beta\alpha} = \iint\{[(\sigma_{rr}^{(\alpha)}\cos(n,\ r) + \sigma_{r\theta}^{(\alpha)}\cos(n,\ \theta) + \sigma_{r3}^{(\alpha)}\cos(n,\ x_3)]\,u_r^{(\beta)} +$$

$$+ [\sigma_{r\theta}^{(\alpha)}\cos(n,\ r) + \sigma_{\theta\theta}^{(\alpha)}\cos(n,\ \theta) + \sigma_{\theta 3}^{(\alpha)}\cos(n,\ x_3)]\,u_\theta^{(\beta)} + [\sigma_{r3}^{(\alpha)}\cos(n, r) +$$

$$+ \sigma_{\theta 3}^{(\alpha)}\cos(n,\ \theta) + \sigma_{33}^{(\alpha)}\cos(n,\ x_3)]\,u_3^{(\beta)}\}d\sigma,\quad (\alpha,\ \beta = 1,\ 2,\ 3,\ 4).$$

We can derive the following formulae

$$
(4.1)\quad \begin{cases}
U_{pq} = h \iint\limits_S \left[t_{33}^{(p)} + c_{33} x^{(p)} + c_{34} \dfrac{1}{r} \varphi_{p,\theta} + c_{35} \varphi_{p,r} \right] x^{(q)} \, d\sigma, \\[2ex]
U_{p4} = h \iint\limits_S \left\{ t_{03}^{(p)} r + \left[c_{34} \left(\dfrac{1}{r} \varphi_{,\theta} + r \right) + c_{35} \varphi_{,r} \right] x^{(p)} \right\} d\sigma , \\[2ex]
U_{4q} = h \iint\limits_S \left[t_{33}^{(4)} + c_{34} \left(\dfrac{1}{r} \varphi_{,\theta} + r \right) + c_{35} \varphi_{,r} \right] x^{(q)} \, d\sigma, \\[2ex]
U_{44} = h \iint\limits_S t_{03}^{(4)} r + c_{44} (\varphi_{,\theta} + r^2) + c_{45} r \varphi_{,r}] \, d\sigma, \quad (p,\ q = 1,\ 2,\ 3).
\end{cases}
$$

In this case the strain energy of the beam is given by

$$
\begin{aligned}
W(a_0, b_0, c_0, \tau_0) &= a_0^2 U_{11} + b_0^2 U_{22} + c_0^2 U_{33} + \tau_0^2 U_{44} + \\
&+ a_0 b_0 (U_{12} + U_{21}) + a_0 c_0 (U_{13} + U_{31}) + a_0 \tau_0 (U_{14} + U_{41}) + \\
&+ b_0 c_0 (U_{23} + U_{32}) + b_0 \tau_0 (U_{24} + U_{42}) + c_0 \tau_0 (U_{34} + U_{43})
\end{aligned}
$$

so that $W(a_0, b_0, c_0, \tau_0)$ is a quadratic positive form.

Taking into account (4.1) it is easy to see that the determinant of the problem is equal to the discriminant of this quadratic form up to a non vanishing factor and consequently it is different of zero.

Some remarks. 1. When $s_{13} = s_{23}$ and s_{34}, s_{35}, $s_{36} = 0$ the problem may be solved in a simpler way.

2. The given method is adequate by solving the general problem of the beams [5], [6] namely when the conditions (1.2) are replaced by

$$
\sigma_{rr} \cos (n,\ r) + \sigma_{r\theta} \cos(n,\ \theta) = \sum_{k=0}^{n} \tau_r^{(k)} (r,\ \theta)\, x_3^k \,, \ldots\ldots
$$

where $\tau_r^{(k)} (r,\ \theta)$ ... are given functions.

3. By adequate particularisations, we can derive the results concerning the orthotropic beams.

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PROBLEMA LUI SAINT-VENANT PENTRU BARE CU ANIZOTROPIE CILINDRICĂ GENERALĂ

(Rezumat)

Se consideră o bară cilindrică pentru care legea lui Hooke este dată de (1.5). Pentru o astfel de bară se rezolvă problema lui Saint-Venant definită de (1.1),(1.2), (1.3).

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ASUPRA ÎNCOVOIERII PLĂCILOR TERMOELASTICE

DE

I. GH. IBĂNESCU

În [1] Goldenveizer formulează problema încovoierii plăcilor elastice ca o problemă de elasticitate tridimensională care se rezolvă prin metode iterative, presupunînd că una din dimensiunile corpului tridimensional este mult mai mică ca celelalte două. Pe baza unui proces iterativ numit de bază se rezolvă problema interioară iar problema stratului limită se rezolvă cu ajutorul unui proces numit ajutător. Sumînd soluţiile celor două probleme se obţine o soluţie aproximativă a problemei încovoierii plăcilor elastice. În nota de faţă aplicăm aceiaşi metodă pentru problema încovoierii plăcilor elastice în prezenţa unui cîmp termic antisimetric faţă de suprafaţa mediană şi staţionar. Vom presupune că placa nu are în interior surse de căldură.

Fie o placă omogenă şi izotropă de grosime $2h$. Placa este raportată la un sistem de axe carteziene $Ox^1x^2x^3$ cu axa Ox^3 pe normala la planul median al plăcii. Ecuaţiile feţelor plăcii vor fi date de $x^3 = \pm h$. O placă elastică este supusă încovoierii dacă forţele masice satisfac următoarele condiţii

$$F_\alpha(x^1, x^2, -x^3) = -F_\alpha(x^1, x^2, x^3) \quad (\alpha = 1, 2);$$

$$F_3(x^1, x^2, -x^3) = F_3(x^1, x^2, x^3).$$

O problemă de termoelasticitate este echivalentă cu o problemă de elasticitate cu forţele masice de forma $\gamma \dfrac{\partial T}{\partial x^i}$ $(i = 1, 2, 3)$. Pentru încovoiere trebuie deci ca $\dfrac{\partial T}{\partial x^3}(x^1, x^2, x^3) = \dfrac{\partial T}{\partial x^3}(x^1, x^2, -x^3)$ de unde rezultă că $T(x^1, x^2, -x^3) = -T(x^1, x^2, x^3)$, adică cîmpul termic trebuie să fie antisimetric faţă de planul median al plăcii.

Avem deci de integrat următorul sistem de ecuaţii:

$$\sigma_{ij,j} = 0, \quad (i = 1, 2, 3; j = 1, 2, 3), \tag{1}$$

$$E\varepsilon_{ij} = (1 + \nu)\sigma_{ij} - \nu s\delta_{ij} + E\alpha_T T\delta_{ij}, \tag{2}$$

$$T_{,kk} = 0, \tag{3}$$

unde $s = \sigma_{11} + \sigma_{22} + \sigma_{33}$, E şi ν sînt modulul lui Young şi respectiv coeficientul lui Poisson, iar T este diferenţa de temperatură dintre starea deformată şi starea nedeformată.

Condiţiile pe feţele plăcii sînt date de:

$$\sigma_{33}(x^1, x^2, \pm h) = \pm\frac{1}{2}p(x^1, x^2), \quad \sigma_{13} = \sigma_{23} = 0, \tag{4}$$

$$T(x^1, x^2, \pm h) = \pm\frac{1}{2}q(x^1, x^2). \tag{5}$$

Pe faţa laterală se vor da condiţii compatibile cu incovoierea plăcilor.

Problema interioară. Vom face transformarea $x^3 = h\zeta$ în sistemul de ecuaţii (1)—(3) şi vom obţine:

$$\sigma_{\alpha\beta,\beta} + h^{-1}\sigma_{\alpha 3,\zeta} = 0, \tag{6}$$

$$\sigma_{3\beta,\beta} + h^{-1}\sigma_{33,\zeta} = 0, \tag{7}$$

$$T_{,\beta\beta} + h^{-2}T_{,\zeta\zeta} = 0, \tag{8}$$

$$E(u_{\alpha,\beta} + u_{\beta,\alpha}) = (1 + \nu)\sigma_{\alpha\beta} - \nu s\delta_{\alpha\beta} + E\alpha_T T\delta_{\alpha\beta}, \tag{9}$$

$$\frac{E}{2}(h^{-1}u_{\alpha,\zeta} + u_{3,\alpha}) = (1 + \nu)\sigma_{\alpha 3}, \tag{10}$$

$$Eh^{-1}u_{3,\zeta} = (1 + \nu)\sigma_{33} - \nu s + E\alpha_T T. \tag{10'}$$

Pentru tensiuni, deplasări şi temperatură vom folosi următoarele dezvoltări:

$$\left\{\begin{array}{ll} \sigma_{\alpha\beta} = h^{-(\varkappa+2)}\sum\limits_{s=1}^{S} h^{s-1}\sigma_{\alpha\beta}^{(s)}, & \sigma_{33} = h^{-\varkappa}\sum\limits_{s=1}^{S} h^{s-1}\sigma_{33}^{(s)}, \\ u_\alpha = h^{-(\varkappa+2)}\sum\limits_{s=1}^{S} h^{s-1}u_\alpha^{(s)}, & u_3 = h^{-(\varkappa+3)}\sum\limits_{s=1}^{S} h^{s-1}u_3^{(s)}, \\ \sigma_{\alpha 3} = h^{-(\varkappa+1)}\sum\limits_{s=1}^{S} h^{s-1}\sigma_{\alpha 3}^{(s)}, & T = h^{-(\varkappa+3)}\sum\limits_{s=1}^{n} h^{s-l}T^{(s)}. \end{array}\right. \tag{11}$$

Înlocuim dezvoltările (11) în ecuaţiile (6)—(10) şi egalăm cu zero coeficienţii puterii h^s. Obţinem sistemul:

$$
(12)\quad \begin{cases}
\sigma^{(s)}_{\alpha\beta,\beta} + \sigma^{(s)}_{\alpha 3,\zeta} = 0, \qquad \sigma^{(s)}_{3\beta,\beta} + \sigma^{(s)}_{33,\zeta} = 0, \\
T^{(s)}_{\alpha\alpha} + T^{(s)}_{\zeta\zeta} = 0, \\
E(u^{(s)}_{\alpha,\beta} + u^{(s)}_{\beta,\alpha}) = 2(1+\nu)\,\sigma^{(s)}_{\alpha,\beta} - 2\nu(\sigma^{(s)}_{\gamma\gamma} + \sigma^{(s-2)}_{33})\delta_{\alpha\beta} + 2E\alpha_T T^{(s)}\,\delta_{\alpha\beta}, \\
E(u^{(s)}_{\alpha,\zeta} + u^{(s)}_{3\alpha}) = 2(1+\nu)\,\sigma^{(s-2)}_{\alpha 3}, \\
Eu^{(s)}_{3,\zeta} = (1+\nu)\sigma^{(s-4)}_{33} - \nu(\sigma^{(s-2)}_{\gamma\gamma} + \sigma^{(s-4)}_{33}) + E\alpha_T\, T^{(s-2)}.
\end{cases}
$$

Cantităţile ce apar în relaţiile (12) sînt funcţii numai de x^1, x^2 şi ζ. Toate cantităţile cu $s \leqslant 0$ sînt nule. Sistemului de ecuaţii (12) îi ataşăm sistemul numit omogen obţinut din (12) lăsînd la o parte termenii cu indici mai mici ca s;

$$
(13)\quad \begin{cases}
\sigma^{(s)}_{\alpha\beta,\beta} + \sigma^{(s)}_{\alpha 3,\zeta} = 0; \qquad \sigma^{(s)}_{3\beta,\beta} + \sigma^{(s)}_{33,\zeta} = 0, \quad T^{(s)}_{\zeta\zeta} = 0, \\
E(u^{(s)}_{\alpha,\beta} + u^{(s)}_{\beta,\alpha}) = 2(1+\nu)\,\sigma^{(s)}_{\alpha\beta} - 2\nu\sigma^{(s)}_{\gamma\gamma}\,\delta_{\alpha\beta} + 2E\alpha_T\, T^{(s)}\,\delta_{\alpha\beta}, \\
E(u^{(s)}_{\alpha,\zeta} + u^{(s)}_{3,\alpha}) = 0, \qquad Eu^{(s)}_{3\,\zeta} = 0.
\end{cases}
$$

Soluţia generală a sistemului (13) este dată de:

$$
(14)\quad \begin{cases}
u^{(s)}_3 = W^{(s)}_0(x^1, x^2), \qquad u^{(s)}_\alpha = -\zeta W^{(s)}_{0,\alpha}\; T^{(s)} = T^{(s)}_0\,\zeta, \\
\sigma^{(s)}_{\alpha\beta} = -\dfrac{E\zeta}{1+\nu} W^{(s)}_{0,\alpha\beta} - \dfrac{\nu E}{1-\nu^2}\,\zeta\Delta W^{(s)}_0\,\delta_{\alpha\beta} - \dfrac{\zeta E\alpha_T}{1-\nu} T^{(s)}_0\,\delta_{\alpha\beta}, \\
\sigma^{(s)}_{\alpha 3} = \dfrac{\zeta^2}{2}\left(\dfrac{E}{1+\nu} W^{(s)}_{0,\alpha\gamma\gamma} + \dfrac{\nu E}{1-\nu^2} W^{(s)}_{0,\alpha\gamma\gamma} + \dfrac{E\alpha_T}{1-\nu} T^{(s)}_{0,\alpha}\right) + \sigma^{(s)}_{\alpha 30}, \\
\sigma^{(s)}_{33} = -\dfrac{\zeta^3}{6}\left(\dfrac{E}{1+\nu}\Delta\Delta W^{(s)}_0 + \dfrac{\nu E}{1-\nu^2}\Delta\Delta W^{(s)}_0 + \dfrac{E\alpha_T}{1-\nu}\Delta T^{(s)}_0\right) - \zeta\sigma^{(s)}_{\alpha 30,\alpha}, \\
\text{unde} \qquad \Delta = \dfrac{\partial^2}{\partial x^{12}} + \dfrac{\partial^2}{\partial x^{22}},
\end{cases}
$$

în care $W^{(s)}_0$ şi $T^{(s)}_0$ sînt funcţii numai de două variabile x^1, x^2.

Soluţia particulară a sistemului neomogen este dată de :

$$
(15)\quad \begin{cases}
E u_3^{*(s)} = \int\limits_0^{\zeta} [(1+\nu)\,\sigma_{33}^{(s-4)} - \nu(\sigma_{\gamma\gamma}^{(s-2)} + \sigma_{33}^{(s-4)}) + E\alpha_T\, T^{(s-2)}]\mathrm{d}\zeta, \\[2ex]
E u_\alpha^{*(s)} = \int\limits_0^{\zeta} [-E u_{3,\alpha}^{*(s)} + 2(1+\nu)\,\sigma_{\alpha 3}^{(s-2)}]\,\mathrm{d}\zeta, \\[2ex]
T^{*(s)} = -\int\limits_0^{\zeta} \mathrm{d}\zeta \int\limits_0^{\zeta} \Delta T^{(s-2)}\,\mathrm{d}\zeta, \\[2ex]
\sigma_{\alpha\beta}^{*(s)} = \dfrac{E}{2(1+\nu)}(u_{\alpha,\beta}^{*(s)} + u_{\beta,\alpha}^{*(s)}) + \dfrac{\nu E}{1-\nu^2}\, u_{\gamma\gamma}^{*(s)}\,\delta_{\alpha\beta} + \\[2ex]
\qquad + \dfrac{\nu}{1-\nu}\,\sigma_{33}^{(s-2)}\,\delta_{\alpha\beta} - \dfrac{E\alpha_T}{2(1-\nu)}\, T^{*(s)}\,\delta_{\alpha\beta}, \\[2ex]
\sigma_{\alpha 3}^{*(s)} = -\int\limits_0^{\zeta} \sigma_{\alpha\beta,\beta}^{*(s)}\,\mathrm{d}\zeta, \qquad \sigma_{33}^{*(s)} = -\int\limits_0^{\zeta} \sigma_{\beta 3,\beta}^{*(s)}\,\mathrm{d}\zeta.
\end{cases}
$$

Toate cantităţile stelate cu indici $s = 1$ şi $s = 2$ sînt nule. Soluţia generală a sistemului omogen o notăm cu $u_{3i}^{(s)}$, $u_{\alpha i}^{(s)}$, $T_{\alpha i}^{(s)}$, $\sigma_{\alpha\beta i}^{(s)}$ $\sigma_{\alpha 3 i}^{(s)}$, $\sigma_{33i}^{(s)}$. Soluţia generală a sistemului (12) va fi dată de :

$$
(16)\quad \begin{cases}
u_3^{(s)} = u_{3i}^{(s)} + u_3^{*)s)}, & \sigma_{\alpha\beta}^{(s)} = \sigma_{\alpha\beta i}^{(s)} + \sigma_{\alpha\beta}^{*(s)}, \\
\sigma_{\alpha 3}^{(s)} = \sigma_{\alpha 3 i}^{(s)} + \sigma_{\alpha 3}^{*(s)}, & \sigma_{33}^{(s)} = \sigma_{33 i}^{(s)} + \sigma_{33}^{*(s)}, \\
u_\alpha^{(s)} = u_{\alpha i}^{(s)} + u_\alpha^{*(s)}, & T^{(s)} = T_i^{(s)} + T^{*(s)}.
\end{cases}
$$

Condiţiile pe feţe vor fi îndeplinite dacă :

$$
(17)\quad \begin{cases}
\sigma_{33}^{(1)}(x^1, x^2, \pm 1) = \pm p(x^1, x^2), \\
\sigma_{\alpha 3}^{(1)}(x^1, x^2, \pm 1) = 0, \\
T^{(1)}(x^1, x^2, \pm 1) = \dfrac{1}{2}\, q(x^1, x^2),
\end{cases}
$$

iar pentru $s > 1$, $\sigma_{33}^{(s)} = \sigma_{\alpha 3}^{(s)} = T^{(s)} = 0$.

Rezultă deci

$$h^{-\varkappa}(\sigma_{33}^{(1)} + \overset{*}{\sigma}{}_{33}^{1}) = \pm \frac{1}{2} p(x^1, x^2), \qquad \overset{*}{\sigma}{}_{33}^{1} = 0. \tag{18}$$

Deoarece $p(x^1, x^2)$ nu depinde de h va trebui să alegem $\varkappa = 0$. Pentru $s = 1$ rezultă ecuaţia

$$\Delta\Delta W_0^{(1)} + E\alpha_T (1 + \nu) \Delta T_0^{(1)} = -\frac{3(1 - \nu^2)}{2E} p, \tag{19}$$

pe care trebuie să o satisfacă funcţiile. $W_0^{(1)}$ şi $T_0^{(1)}$

Funcţia $T_0^{(1)}$ este dată de:

$$T_0^{(1)} = \pm \frac{1}{2} q. \tag{20}$$

Ecuaţia (19) este ecuaţia clasică a încovoierii plăcilor termoelastice. Pentru celelalte aproximante rezultă ecuaţii analoge.

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ON THE BENDING OF THERMOELASTIC PLATES

(Summary)

In this paper we start from the equations of the three-dimensional thermoelasticity and by change of variable $x^3 = h\xi$ and developing (11) we get the classical equation of the bending of thermoelastic plates without using Love-Kirchhoff hypothesis.

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FINITE DEFORMATION CLASSES OF ISOTROPIC AND ANISOTROPIC SIMPLE SOLIDS

BY

C. FETECĂU

1. Introduction

In the continuum mechanics of non-linear media, a simple material is defined by the constitutive equation [1]

$$\mathsf{T}(t) = \overset{\infty}{\underset{s=0}{\mathsf{H}}}(\mathsf{F}(t-s)) = \overset{t}{\underset{\sigma=-\infty}{\mathsf{H}}}(\mathsf{F}\,\tau)), \tag{1.1}$$

where H is a functional which assigns to the tensor-valued function $\mathsf{F}(\tau)$ the symmetric tensor $\mathsf{F}(t)$. Here, $\mathsf{F}(\tau)$ is the deformation gradient at time τ, H [1]) is the response functional relative to a local reference configuration and $\mathsf{T}(t)$ is the stress at time t.

The response functional H satisfies the relation

$$\mathsf{Q}(t)\overset{t}{\underset{\sigma=-\infty}{\mathsf{H}}}(\mathsf{F}(\tau))\,\mathsf{Q}(t)^T = \overset{t}{\underset{\sigma=-\infty}{\mathsf{H}}}(\mathsf{Q}(\tau)\,\mathsf{F}(\tau)) \tag{1.2}$$

for every orthogonal tensor function $Q(\tau)$ and every deformation gradient $\mathsf{F}(\tau)$. By definition, the isotropy group g of a simple material is the set of all unimodular tensors H, such that,

$$\overset{t}{\underset{\sigma=-\infty}{\mathsf{H}}}(\mathsf{F}(\tau)) = \overset{t}{\underset{\sigma=-\infty}{\mathsf{H}}}(\mathsf{F}(\tau)\,\mathsf{H}) \tag{1.3}$$

for all invertible tensors $\mathsf{F}(\tau)$.

---

[1]) The dependence of H on the local configuration has been suppressed in the notation.

We recall that an orthogonal tensor Q belongs to the isotropy group of a simple material if and only if [1]

$$\mathsf{Q} \underset{\sigma=-\infty}{\overset{t}{\mathsf{H}}} (\mathsf{F}(\tau))\,\mathsf{Q}^T = \underset{\sigma=-\infty}{\overset{t}{\mathsf{H}}} (\mathsf{Q}\mathsf{F}(\tau)\,\mathsf{Q}^T) \tag{1.4}$$

holds identically in $F(\tau)$.

The purpose of the present paper is to find deformation classes [2]) of the simple solids and the corresponding stress fields, without simplifications imposed on the constitutive equations.

The equations of motion are given in Sec. **2** and some properties of the orthogonal tensors belonging to the isotropy group are obtained in Sec. **3**. The Sections **4**,**5** contain some results concerning the stress fields for various deformation classes of homogeneous isotropic and anisotropic solids.

2. Equations of Motion

The field equations for a material subjected to a body force per unit mass, are

$$T_{ij,j} + \rho b_i = \rho a_i\,, \qquad T_{ij} = T_{ji}, \tag{2.1}$$

where the suffixes denote components in the rectangular Cartesian frame $(x,\ y,\ z)$ and the comma denotes differentiation with respect to the corresponding coordinates. Here ρ is the density of the material and $\mathbf{a}$ is the acceleration. If assume the body force $\mathbf{b}$ to be conservative with single-valued potential φ

$$\mathbf{b} = \operatorname{grad}\varphi\,, \tag{2.2}$$

then, the equations of motion become

$$T_{ij,j} + \rho\varphi_{,i} = \rho a_i. \tag{2.3}$$

We consider here, the deformations of solids for which

$$a_i = -\,\xi_{,i}\,, \tag{2.4}$$

where ξ is a single-valued function (see [2] or [3]).

In this case, the equations of motion (2.3) become

$$T_{ij,j} + \rho\psi_{,j} = 0\,, \tag{2.5}$$

where $\psi = \varphi + \xi$. Obviously, from (2.4) it obtains

$$\operatorname{curl}\mathbf{a} = 0\,. \tag{2.6}$$

---

[2]) Here, by a class of deformation we understand the set of deformations for which the deformation gradient is of the same form.

3. On the Orthogonal Tensors Belonging to the Isotropy Group

Let Q be an orthogonal tensor belonging to the isotropy group g of a simple material and let $\mathsf{F}(\tau)$ be a deformation gradient satisfying

$$\mathsf{F}(\tau)\,\mathsf{Q} = \mathsf{Q}\mathsf{F}(\tau). \tag{3.1}$$

From (1.4) we infer that

$$\underset{\sigma=-\infty}{\overset{t}{\mathsf{H}}}(\mathsf{F}(\tau))\,\mathsf{Q} = \mathsf{Q}\underset{\sigma=-\infty}{\overset{t}{\mathsf{H}}}(F(\tau)) \tag{3.2}$$

and hence, the appropriate stress tensor satisfies

$$\mathsf{T}\mathsf{Q} = \mathsf{Q}\mathsf{T}, \qquad \mathsf{T} = \mathsf{T}(t). \tag{3.3}$$

Let $\mathsf{F}(\tau) = \mathsf{R}(\tau)\,\mathsf{U}(\tau)$ be the polar decomposition of $\mathsf{F}(\tau)$, [1, Sec. **23**].

P r o p o s i t i o n. *An unimodular tensor* H *belongs to the isotropy group of a simple material if and only if*

$$\underset{\sigma=-\infty}{\overset{t}{\mathsf{H}}}(\mathsf{U}(\tau)) = \underset{\sigma=-\infty}{\overset{t}{\mathsf{H}}}(\mathsf{U}(\tau)\,\mathsf{H}) \tag{3.4}$$

holds identically in $\mathsf{U}(\tau)$.

P r o o f. Necessity is obviously. The identity (1.3) being true for every $\mathsf{F}(\tau)$, in particular, it is true also for $\mathsf{F}(\tau) = \mathsf{U}(\tau)$.

For sufficiency, we multiply (3.4) by $\mathsf{R}(t)$ (where $\mathsf{R}(t)$ is an arbitrary orthogonal tensor function) and use (1.2).

From this proposition and the relation (1.2), it results that an orthogonal tensor belongs to the isotropy group g if and only if

$$\underset{\sigma=-\infty}{\overset{t}{\mathsf{H}}}(\mathsf{Q}\mathsf{U}(\tau)\,\mathsf{Q}^T) = \mathsf{Q}\underset{\sigma=-\infty}{\overset{t}{\mathsf{H}}}(\mathsf{U}(\tau))\,\mathsf{Q}^T \tag{3.5}$$

holds identically in $\mathsf{U}(\tau)$.

Let $\mathsf{F}(\tau)$ be a deformation gradient for which

$$\mathsf{U}(\tau)\,\mathsf{Q} = \mathsf{Q}\mathsf{U}(\tau), \tag{3.6}$$

where Q is an orthogonal tensor belonging to g. From (3.5), it results

$$\underset{\sigma=-\infty}{\overset{t}{\mathsf{H}}}(\mathsf{U}(\tau))\,\mathsf{Q} = \mathsf{Q}\underset{\sigma=-\infty}{\overset{t}{\mathsf{H}}}(\mathsf{U}(\tau)) \tag{3.7}$$

and respectively for the stress tensor

---

[3]) In order to avoid tedious repetition, the word „solid" will hencefort be used to mean „simple solid".

$$\mathsf{TA} = \mathsf{AT}, \tag{3.8}$$

where $\mathsf{A} = \mathsf{RQR}^T$ and $\mathsf{R} = \mathsf{R}(t)$.

The remainder of this paper is concerned with deformations of homogeneous, isotropic and anisotropic solids. In the following, the solid will be refered to an undistorted state [1]. In this case, it is easily seen that g is the direct product of the two-element group $\{-1,1\}$ and a group g_0 which consists only of proper orthogonal transformations (i. e. rotations). For our considerations, we need only the generators of g_0 which are given in [1, Sec. 33].

4. Isotropic Solids

Let us consider the class of deformations given by

$$\mathsf{F}(\tau) = f(\tau)\,\mathsf{R}(\tau), \qquad \mathsf{R}(\tau) \in o. \tag{4.1}$$

From (3.6) and (3.8), it results that the corresponding stress should be hydrostatic i.e.,

$$\mathsf{T} = p\,\mathsf{1}. \tag{4.2}$$

Since, every conformal deformation carries an undistorted state into another else, the formula (4.2) shows that the stress calculated for an undistorted state of un isotropic solid refered to another undistorted state as a reference configuration, is hydrostatic.

From the previous conclusion it is easily seen that the stress on an undistorted state of an isotropic solid, is hydrostatic. For the elastic materials this result has been obtained in [1].

As an example, the deformation

$$\begin{cases} x = A(t)\,X + B(t), \\ y = A(t)\,Y + C(t), \\ z = A(t)\,Z + D(t), \end{cases} \tag{4.3}$$

where (X, Y, Z) and (x, y, z) are Cartesian co-ordinates in the reference and loaded configurations, respectively, satisfies (2.6) and (4.1).

From (1.1) and (2.5), it results that p and ψ are arbitrary functions [4]) of t.

5. Anisotropic Solids

a) *The monoclinic solids.* Let us consider the class of deformations given by

[4]) We use the term of arbitrary function, but the property of differentiability is assumed.

$$(5.1)\qquad [\mathsf{F}(\tau)] = \begin{bmatrix} F_{11}(\tau) & F_{12}(\tau) & 0 \\ F_{21}(\tau) & F_{22}(\tau) & 0 \\ 0 & 0 & F_{33}(\tau) \end{bmatrix}.$$

From (3.1) and (3.3), it results that the corresponding stress tensor is given by

$$(5.2)\qquad [\mathsf{T}] = \begin{bmatrix} T_{11} & T_{12} & 0 \\ T_{12} & T_{22} & 0 \\ 0 & 0 & T_{33} \end{bmatrix}$$

and (2.5) is satisfied, if and only if,

$$(5.3)\qquad \begin{cases} T_{11} = -\rho\psi + \Psi_{,y}, \quad T_{12} = -\Psi_{,x}, \\ T_{22} = -\rho\psi + \int \Psi_{,xx}\, dy + \chi, \\ T_{33} = -\rho\psi + \Phi, \end{cases}$$

where Φ, χ and Ψ are arbitrary functions of (x, y, t), (x, z, t) and (x, y, z, t), respectively.

b) *The rhombic solids.* In the same way as in the case of the monoclinic solids, it results that the corresponding stresses for the classes of deformations

$$(5.4)\qquad \begin{bmatrix} F_{11}(\tau) & 0 & 0 \\ 0 & F_{22}(\tau) & 0 \\ 0 & 0 & F_{33}(\tau) \end{bmatrix}, \begin{bmatrix} F_{11}(\tau) & 0 & 0 \\ 0 & F_{22}(\tau) & F_{23}(\tau) \\ 0 & F_{32}(\tau) & F_{33}(\tau) \end{bmatrix}, \begin{bmatrix} F_{11}(\tau) & 0 & F_{13}(\tau) \\ 0 & F_{22}(\tau) & 0 \\ F_{31}(\tau) & 0 & F_{33}(\tau) \end{bmatrix}$$

must be, respectively, of the forms

$$(5.5)\qquad \begin{bmatrix} T_{11} & 0 & 0 \\ 0 & T_{22} & 0 \\ 0 & 0 & T_{33}, \end{bmatrix}, \begin{bmatrix} T_{11} & 0 & 9 \\ 0 & T_{22} & T_{23} \\ 0 & T_{23} & T_{33} \end{bmatrix}, \begin{bmatrix} T_{11} & 0 & T_{13} \\ 0 & T_{22} & 0 \\ T_{13} & 0 & T_{33} \end{bmatrix}.$$

It is easy to see that (2.6) is satisfied for all deformations from the class $(5.4)_1$.

c) *The tetragonal and hexagonal solids.* We consider the class of deformations of the form

$$(5.6)\qquad [\mathsf{F}(\tau)] = \begin{bmatrix} F_{11}(\tau) & F_{12}(\tau) & 0 \\ F_{12}(\tau) & F_{11}(\tau) & 0 \\ 0 & 0 & F_{33}(\tau) \end{bmatrix}.$$

This class satisfies (3.1) for every $\mathsf{Q} \in g$, g being the isotropy group of a solid of the type 4,8 or 10 ([1, Sec. **33**]). From (3.3) we obtain

$$[\mathsf{T}] = \begin{bmatrix} T_{11} & 0 & 0 \\ 0 & T_{11} & 0 \\ 0 & 0 & T_{33} \end{bmatrix} \tag{5.7}$$

or

$$\mathsf{T} = -p\,\mathsf{1} + q\,\mathsf{k} \otimes \mathsf{k}, \quad p = -T_{11}, \quad q = T_{33} + T_{11} \tag{5.8}$$

i. e. the deformations from the class (5.6) are maintained under an uniaxial tension in the direction of k superimposed upon an ambient pressure p.

For the solids belonging to the types 5,9 and 11 we obtain that the stresses corresponding to the classes of deformations (5.6) and

$$[\mathsf{F}(\tau)] = \begin{bmatrix} F_{11}(\tau) & 0 & 0 \\ 0 & F_{11}(\tau) & 0 \\ 0 & 0 & F_{33}(\tau) \end{bmatrix} \tag{5.9}$$

are given by the same formula (5.7).

From the relations (3.1) and (3.3) we conclude that the stress on an undistorted state of a solid belonging to the tetragonal or hexagonal solids, should be of the form (5.7) [5]).

Remark. By the virtue of the preceding conclusion and (5.6), (5.9) we can observe that the stress field (5.7) does not characterize the undistorted states of the tetragonal or hexagonal solids.

The most general deformations from the class (5.9) are given by

$$\begin{cases} x = A(t)\,X + B(t), \\ y = A(t)Y + B(t), \\ z = W(Z, t), \end{cases} \tag{5.10}$$

where A, B, C and W are arbitrary functions. For these deformations, the relation (2.6) is satisfied. From (1.1) and (2.5) we have

$$\begin{cases} T_{11} = \Phi(z, t), \\ T_{33} = -\rho\psi(z, t) + \Psi(t). \end{cases} \tag{5.11}$$

d) *The cubic solids.* For this type of solids as for isotropic solids, the stress corresponding to the class (4.1) must be hydrostatic. The results established in Sec. 4 remain also valid for these solids.

Also, it is easily seen that, if $\mathsf{F}(\tau)$ is given by (5.4) or (5.6) then T have the forms corresponding to the respective systems.

Similarly, for the classes of deformations

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[5]) The similar result for the corresponding elastic solids has been obtained in [1]

$$(5.12)\quad \begin{bmatrix} F_{11}(\tau) & 0 & 0 \\ 0 & F_{22}(\tau) & F_{23}(\tau) \\ 0 & -F_{23}(\tau) & F_{22}(\tau) \end{bmatrix}, \begin{bmatrix} F_{11}(\tau) & 0 & F_{13}(\tau) \\ 0 & F_{22}(\tau) & 0 \\ -F_{13}(\tau) & 0 & F_{11}(\tau) \end{bmatrix} \text{ or } \begin{bmatrix} F_{11}(\tau) & F_{12}(\tau) & F_{13}(\tau) \\ F_{13}(\tau) & F_{11}(\tau) & F_{12}(\tau) \\ F_{12}(\tau) & F_{13}(\tau) & F_{11}(\tau) \end{bmatrix},$$

the corresponding stress-tensors are given, respectively, by

$$(5.13)\quad \begin{bmatrix} T_{11} & 0 & 0 \\ 0 & T_{22} & 0 \\ 0 & 0 & T_{22} \end{bmatrix}, \begin{bmatrix} T_{11} & 0 & 0 \\ 0 & T_{22} & 0 \\ 0 & 0 & T_{11} \end{bmatrix} \text{ and } \begin{bmatrix} T_{11} & T_{12} & T_{12} \\ T_{12} & T_{11} & T_{12} \\ T_{12} & T_{12} & T_{11} \end{bmatrix}.$$

The last tensor T from (5.13) can be writen in the form

$$(5.14)\quad \mathsf{T} = T_{11}\,\mathsf{1} + T_{12}\,\mathsf{M},$$

where

$$(5.15)\quad \mathsf{M} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}.$$

The equations of motion (2.5) shall be satisfied if and only if

$$(5.16)\quad \begin{cases} T_{11} = -\rho\psi + g\,(x-y,\ \ y-z,\ \ t) + \dfrac{2}{3}\,h(x+y+z,\ t), \\ T_{12} = g(x-y,\ y-z,\ t) - \dfrac{1}{3}\,h(x+y+z,\ t), \end{cases}$$

where g and h are arbitrary functions of arguments.

We consider a particular deformation from the class (5.13).

$$(5.17)\quad \begin{cases} x = A(t)X + B(t)Y + C(t)Z + D(t), \\ y = C(t)X + A(t)Y + B(t)Z + E(t), \\ z = B(t)X + C(t)Y + A(t)Z + F(t). \end{cases}$$

In this case, from (1.1) and (2.5) it results that T_{11}, T_{12} and ψ must be constants with respect to x, y, z. The condition (2.6) implies for the functions A, B and C the following relation

$$(5.18)\quad \ddot{A}\,(B-C) + \ddot{B}\,(C-A) + \ddot{C}\,(A-B) = 0.$$

The relation (5.18) is satisfied, for example when two of them are equals or all are linears.

e) *Transversely istropic solids*. For this solids, it follows easily that the stress tensors corresponding to (5.1) and (5.6) are given, respectively, by (5.2) and (5.7).

Deformations for which $\mathsf{F}(\tau)$ is of the form (5.6), have been studied in [4] for transversely isotropic elastic solids, but, the results can be readly extended. All results of the present section remain true for the isotropic solids too.

It seems that many results established for solids can be obtained in a similar way for the simple fluids. We mention that from (1.2) can be found other restrictions on the stress tensor components.

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CLASE DE DEFORMĂRI FINITE ALE SOLIDELOR SIMPLE ISOTROPE ŞI ANISOTROPE

(Rezumat)

Scopul articolului e să găsească clase de deformări ale solidelor simple, ce pot fi studiate fără simplificări ale ecuaţiilor constitutive. Se folosesc principiile obiectivităţii şi simetriei.

Pentru solidele simple isotrope şi cubice se foloseşte o definiţie echivalentă a grupului de isotropie. Se găseşte că unele rezultate cunoscute pentru solide simple elastice se păstrează şi în cazul solidelor simple.

Rezultatele obţinute în § 5 pentru solide anizotrope rămîn adevărate şi pentru solide isotrope. În acest caz vom avea noi restricţii provenite din principiul simetriei.

Toate rezultatele obţinute aici se pot aplica cu uşurinţă şi la fluide simple.

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SUR LE PROBLÈME D'ÉTUDE DES CHAMPS D'ACCÉLÉRATIONS D'ORDRE QUELCONQUE DANS LE MOUVEMENT PLAN DES CHAINES CINÉMATIQUES DES SOLIDES RIGIDES

PAR

C. HUIU

La parution il y a quelques années grâce à l'innitiative du Département de Mécanique de l'Université Technique d'Eindhoven des volumes de „Bibliographie de Cinématique" [1], contanant approximativement 7.000 sourses qui datent de 1525, 1969 y inclus, a été pour moi non seulement d'une aide précieuse lors de l'élaboration de ma Thèse de doctorat [2], mais aussi dans nos recherches actuelles portant sur quelques domaines de Mécanique rationnelle et appliquée.

Dans ce qui suit on donne, dans le cadre d'un nouveau approfondissement de l'étude des champs d'accélérations d'ordre quelquonque des éléments des mécanismes plans supposés constitués des corps solides rigides, qui se situe, entre autres, dans l'ensemble des effors de renouvellement de l'entière Mécanique classique, inaugurée à Jassy [3], [4] et cristallisés d'or et déjà dans toute une série de volumes de „Mécanique pour l'Ingénieur" [5], le suivant.

Théorème 1. *Dans le mouvement de régime permanent de la manivelle d'un système bielle-manivelle plan ayant les longueurs de celles-ci égales entre elles, le lieu géométrique des points $M(x', y')$ appartenant au plan de la bielle et dont les valeurs des accélerations d'ordre pair $(2j-2)$ et impair $(2k-1)$ quelconques sont des nombres positifs constants, à savoir C_{2j-2} et C_{2k-1}, respectivement, est une circonférence qui a pour centre la tête de la bielle et pour carré du rayon*

$$\frac{1}{2}\left(C_{2k-1}^2\,\dot{\theta}^{4k} + C_{2j-2}^2\,\dot{\theta}^{(4j-2)} - r^2\right),$$

ou $\dot{\theta} = \mathrm{d}\theta/\mathrm{d}t$ est la vitesse angulaire constante de la manivelle et r sa longueur (égale a celle de la bielle).

Dans le cas général de mouvement du système bielle-manivelle plan considéré, on obtient le lieu géométrique cherché, à la suite de la programmation adéquate aux computers, des équations ci-dessous :

$$\begin{cases} (A_{2k-1}^2+B_{2k-1}^2)(x'^2+y'^2)+4r(A_{2k-1}\sin\theta-B_{2k-1}\cos\theta)(A_{2k-1}\cos\theta+ \\ +B_{2k-1}\sin\theta)x'+4r(r-y')(A_{2k-1}\cos\theta+B_{2k-1}\sin\theta)^2=C_{2k-1}^2, \\ (A_{2j-2}^2+B_{2j-2}^2)(x'^2+y'^2)+4r(A_{2j-2}\sin\theta-B_{2j-2}\cos\theta)(A_{2j-2}\cos\theta+ \\ +B_{2j-2}\sin\theta)x'+4r(r-y')(A_{2j-2}\cos\theta+B_{2j-2}\sin\theta)^2=C_{2j-2}^2, \end{cases}$$

ou les coefficients A_{n+1} *et* B_{n+1} *sont donnés par les formules de recurrence* [6]—[7] :

$$\begin{cases} A_{n+1}=\dot{A}_n+\dot{\theta}B_n, \quad B_{n+1}=\dot{B}_n+\dot{\theta}A_n, \quad (n=1,2,\ldots) \\ A_1=\dot{\theta}^2, \quad B_1=\ddot{\theta}, \quad \left(\cdot\equiv\dfrac{d}{dt}\right). \end{cases}$$

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CONTRIBUŢII LA STUDIUL CÎMPURILOR DE ACCELERAŢII DE ORDIN OARECARE ÎN MIŞCAREA PLANĂ A LANŢURILOR CINEMATICE CONSTITUITE DIN SOLIDE RIGIDE

(Rezumat)

Ţinînd seama de bogatul material informativ privitor la domeniul Cinematicii din [1], în care sînt citate, între altele, mai multe lucrări ale specialiştilor din ţară şi îndeosebi din Şcoala de Mecanică aplicată din Iaşi şi avînd în vedere ansamblul rezultatelor proprii expuse în teza de doctorat [2], în această notă se prezintă o teoremă referitoare la cîmpuri de accelerații de ordin oarecare pentru punctele planului mobil invariabil legat de biela unui mecanism manivelă-piston.

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L'ABSORBTION RÉSONANTE AUX COUCHES MINCES ÉLECTROLYTIQUES DE Ni ET Co AVEC ANISOTROPIE UNIAXE[1])

PAR

R. GRIMBERG et C. NAUM

1. Introduction

La résonance ferromagnétique aux couches minces est une méthode d'etude par laquelle ont peut déterminer leurs propriétés magnétiques. Mais l'existence de l'anisotropie uniaxe permet, en rapport avec le massif, l'emploi de la résonance ferromagnétique dans le domaine des basses fréquences aussi — MHz.

À l'encontre d'autres travaux publiés antérieurement, nous *y* avon étudié la résonance ferromagnétique de basse fréquence aux couches minces magnétostrictives, le travail présent étant le premier parru qui aborde un tel problème.

2. La préparation des échantillons et l'installation de mesure

Les couches minces policristallines de Ni et Co déposées électrolytique sur des disques en or et en platine de 13 mm diamètre ont été préparées d'apres les récettes donées en [1]. L'épaisseur des couches a été de 1 000 Å déterminée à l'aide d'un microscope interférentiel.

L'instalation de mesure employée a été décrite en [2], [3] étant constituée principallement d'un autooscillateur Pound modifié pour obtenir une stabilité en fréquence de 10^{-4} et 10^{-2} en amplitude. La sensibilité du montage est de 10^{19} spins DPPH quand la bande de passage a 50 KHz. La fréquence peut être variée entre 3 et 10 MHz sans actioner la bobine de sonde. La modulation est de typ Zeemann à la fréquence de 50 Hz, la

[1]) Présenté à la Session scientifique jubiliaire de la Faculté d'Électrotechnique de Jassy, le 21—23 décembre 1972.

stabilité en amplitude et en fréquence du champ modulateur étant 10^{-2}. Le mésurement du champ critique de déplacement des parois a été fait utilisant la méthode de la susceptibilité différentielle de Feldtkeller [4], adaptée aux conditions des mesurements en h. f. On a tenu compte des restrictions indiquées en [5]. Les erreures de mesure sur le champ critique ont été inférieures à 5%.

3. Dates expérimentales et leur interprétation

Le champ critique de rotation de l'aimantation, qui pour des fréquences très basses coïncide avec le champ d'anisotropie et le champ critique de déplacement des parois, qui dans les même conditions coïncide avec le champ coercitif de l'échantillon, varient avec la fréquence employée et avec l'amplitude du champ de h. f.

Dans les fig. 1 et 2 nous présentons ces deux variations pour les deux champs critiques.

De la fig. 1 on peut conclure que le champ critique de rotation, qui correspond au champ de résonance, varie rélativement peu en fonction de l'amplitude et de la fréquence du champ de h. f. Le champ de résonance diffère de champ d'anisotropie par un facteur aditif, conformement à la rélation [6] :

$$H_0 = H_k + \frac{1}{15A}\left(\frac{D^8K^8d^2}{M_s}\right)^{1/5},$$

où A — la constante d'échange, D — le diamètre moyen des cristallites, K — la constante d'anisotropie magnétocristalline, d — l'épaisseur de la couche mince, M_s — l'aimantation à saturation.

La platine et l'or employé comme support ont le même système de cristallisation Fm 3m, mais les constantes des réseaux sont différentes : 3,923 Å pour Pt et 4,079 Å pour Au. Comme les dimensions des cristallites de la couche dépendent aussi de la constante du réseau du support, même si celui-ci est polycristallin, le champ d'anisotropie des couches déposées sur Au différera du champ de résonance mesuré dans le cas des couches déposées sur Pt.

Parce que dans les expériences de résonance ferromagnétique de h. f. les valeurs de la fréquence et de l'amplitude du champ de h. f. sont petites, on peut affirmer avec une bonne précision que les valeurs mesurées représentent le champ d'anisotropie même.

Le champ critique de déplacement des parois est fort influencé tant de l'amplitude que de la fréquence du champ de h. f., fait qui peut être remarqué dans la fig. 2.

Une diminution nette du champ critique est évidente quand l'amplitude du champ de h. f. et la fréquence de travail augmentent.

On n'a pas réussi à obtenir, dans les conditions expérimentales décrites, la résonance des parois Bloch quoique les fortes tensions internes existentes dans les couches conduisent à une plus grande mobilité des parois de 90° dans les domaines marginals.

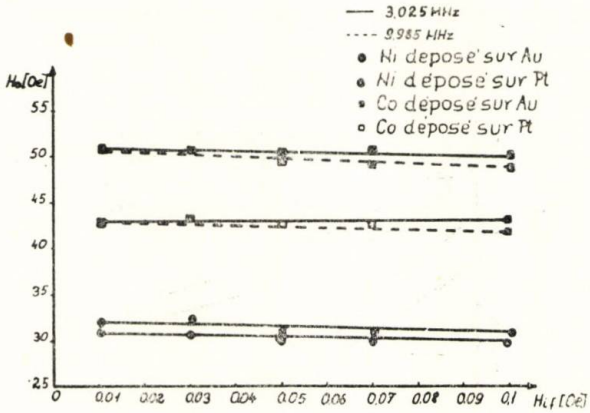

Fig. 1

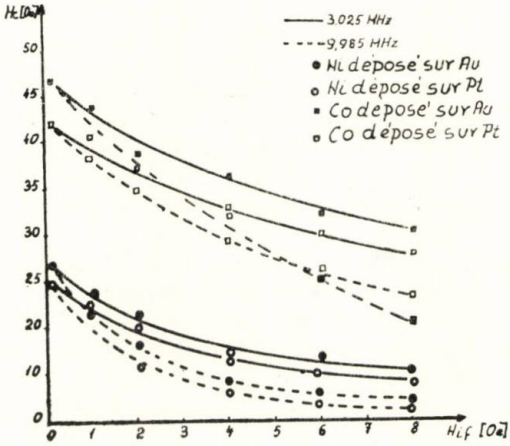

Fig. 2

La largeur de la ligne de résonance peut être écrite comme une somme de deux termes :

$$\Delta H = \Delta H_0 + \frac{2\alpha\omega_0}{\gamma} ,$$

où ΔH_0 est le terme indépendent de la fréquence et conditioné par la dispersion angulaire et en amplitude de l'anisotropie, α — et le facteur d'amortissement de l'équation Gilbert, ω_0 — la pulsation de résonance, γ — le facteur giromagnétique.

Les auteurs ont étudié comme le facteur d'amortissment dépend de la fréquence de travail. Le graphique de cette dépendence est présenté dans la fig. 3.

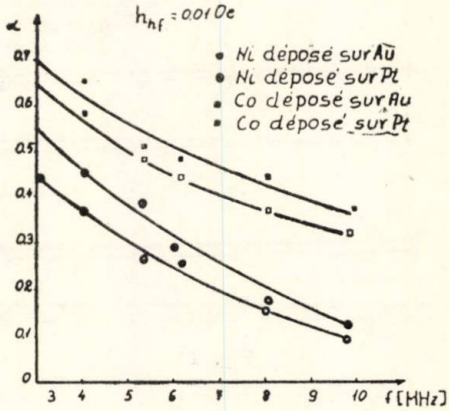

Fig. 3

On voit que le facteur d'amortissement diminue quand la fréquence de travail croit, les plus petits valeurs correspondant aux couches du Ni déposées sur Pt et les plus grandes aux couches du Co déposées sur Au. Comparativement à ces résultats en [2] on a obtenu pour des couches en permalloy (80% Ni), dépaisseur 3 000 Å, un facteur d'amortissement de 0,1 à la fréquence de 3 MHz.

Les dates présentées conduisent à la conclusion que le facteur d'amortissement est en dépendence d'inverse proportionalité avec le grandeur des tensions internes existentes dans la couche. Yl en résulte que les tensions internes influencent tant le terme H_0, par l'intermédiaire des dispersions de l'anisotropie, que le terme $2\alpha\omega_0/\gamma$, grace à α.

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ABSORBŢIA REZONANTĂ LA PĂTURI SUBŢIRI ELECTROLITICE DE Ni şi Co CU ANIZOTROPIE UNIAXIALĂ

(Rezumat)

Se studiază rezonanţa feromagnetică de joasă frecvenţă la pături subţiri electrolitice de Ni şi Co, urmărindu-se variaţia cîmpurilor de prag şi a factorului de amortizare a lui Gilbert în funcţie de condiţiile de depunere şi de rezonanţă.

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LES CARACTÉRISTIQUES D'HYSTÉRÉSIS DE L'ALLIAGE Fe-3,27% Si LAMINÉ ET TRAITÉ THERMOMAGNÉTIQUEMENT

PAR

GH. CĂLUGĂRU, C. CUCIUREANU et ŞT. NOVAC

1. Introduction

Par le processus de laminage au froid de l'alliage Fe-Si, suivi d'un traitement au chaud (récristalisation) on développe dans l'échantillon une texture (110) [001]. Ca veut dire que la direction de facile aimantation [001] est parallèle à la direction du laminage, tandis que la direction de difficile aimantation [110] a une direction transversale. L'orientation des cristallites peut atteindre 90% [1].

Les propriétés magnétiques de l'alliage Fe-Si peuvent être modifiées par traitement magnétique.

Quelques résultats sur Fe-Si orienté et non orienté ont été donné par Moses et Thompson [2] et Houze et ses collaborateurs [3].

Dans ce travail nous étudions l'influence du traitement thérmomagnétique sur les caractéristiques d'hystérésis de l'alliage Fe-3,27% Si laminé, en fonction de la température de traitement, du temps de traitement et de l'intensite du champ magnétique appliqué.

2. La technique expérimentale

Les caractéristiques d'hystérésis des échantillons ont été mesurées à l'aide d'un ferrotester typ Orion. Les échantillons ont été bandes aux dimensions 0,35 mm × 10 mm × 200 mm. Les mesurements ont été faits au long du laminage. Le traitement thermomagnétique a été fait dans un four à quartz, sous atmosphère d'argon. Les échantillons de Fe-Si employés dans nos études ont eu la composition chimique suivante :

96,623% Fe, 3,27% Si, 0,06% Mn, 0,02% C,
0,012% P, 0,004% S, 0,011% Al.

Les échantillons ont été préparés de la manière suivante : après un laminage au chaud en bandes de 2,5 mm × 7,5 mm, un traitement au 950°C pendant 4 heures suivi d'un réfroidissement lent et finalement un laminage au froid en bande de 0,35 mm épaisseur ; les échantillons tellement obtenus ont été soumis au traitement thermomagnétique.

Pour aprécier comparativement le changement des valeurs des grandeurs mesurées, par traitement thermomagnétique, la majorité des résultats expérimentales ont été exprimés par des rapports :

$$H_c/H_{c_0}\,,\quad B_r/B_{r_0}\,,\quad B_{\max}/B_{\max_0}\,,\quad P_{1T}/P_{1T_0}\,,\quad P_{1,5T}/P_{1,5T_0}$$

où H_{c_0}, B_{r_0}, $B_{\max_0}$, P_{1T_0} et $P_{1,5T}$ sont les valeurs pour échantilons laminés avant le traitement thermomagnétique, tandis que H_c, B_r, $B_{\max}$, P_{1T} et $P_{1,5T}$ sont les valeurs corespondantes aux différents stades de traitement.

3. Resultats expérimentals. Discussions

Les caractéristiques magnétiques des échantillons laminés, pour un champ magnétisant de 400 A/m sont : $H_{c_0} = 150$ A/m, $B_{r_0} = 0{,}044$ T, $B_{\max_0} = 0{,}106$ T ; les pertes par hystérésis à 1 T et à 1,5 T sont : $P_{1T_0} = 39 \times 10^4$ T/m³ et $P_{1,5T_0} = 110 \times 10^4$ T/m³.

3.1. Influence de la température de traitement

Dans le tableau 1 sont données les grandeurs mesurées aux différentes températures de traitement pour un temp de traitement de 2 heures et pour un champ magnétique de 2×10^4 A/m, appliqué durant le traitement. Comme on voit, le champ coercitif pour les échantillons traités à 670°K est supérieur au champ correspondant aux échantillons non traités. De 670°K à 1 070°K, H_c diminue tandis que B_r et $B_{\max}$ augmentent.

Tableau 1

Les caractéristiques magnétiques à différentes températures de traitement : Temps de traitement 2 h, *Champ applique*, $H_{\text{magn}} = 2 \times 10^4$ A/m, *Vitesse de refroidissement* 100° K/h.

Temperature de traitement	$\frac{H_c}{H_{c_0}}$	$\frac{B_r}{B_{r_0}}$	$\frac{B_{\max}}{B_{\max_0}}$	$\frac{P_{1T}}{P_{1T_0}}$	$\frac{P_{1,5T}}{P_{1,5T_0}}$
670°K	1,5	5,45	3,58	0,55	0,58
770°K	1,37	18,36	8,96	0,32	0,44
870 °L	0,92	25,69	12,30	0,21	0,36
970°K	0,74	26,48	13,21	0,13	0,12
1 070°K	0,66	27,18	13,87	0,11	0,12
1 070°K	0,71	19,30	traitement thérmique sans champ		

Une variation similaire du champ coercitif en function de la température de traitement a été obtenue par Chaudhary et Qureshi [4] dans le cas du traitement thérmique de Fe-Si en absence du champ magnétique et ils l'ont expliquée à l'aide de la variations des perturbations dans le réseau.

L'augmentation de l'induction rémanente B_r par traitement thermomagnétique nous allons l'expliquer fondés sur l'idée que par l'application du champ magnétique on facilite le développement d'une texture fort monocomponente à l'axe facile [001] dirrigée au long de la direction du laminage, par la diminution de la dispersion des axes faciles. L'induction rémanente B_r a des valeurs supérieures quand le traitement thermique est fait en présence d'un champ magnétique. Ainsi aux échantillons traités à 1070°K sans champ magnétique nous avons obtenu $H_c/H_{c_0} = 0{,}71$ et $B_r/B_{r_0} = 19{,}3$ comparatif aux valeurs $H_c/H_{c_0} = 0{,}66$ et $B_r/B_{r_0} = 27{,}18$, obtenues pour les échantillons traités en champ magnétique. On voit bien que le champ coercitif a des valeurs inférieures pour les échantillons traités en présence du champ.

Quant aux pertes d'énergie par hystérésis, celles-ci diminuent avec le haussement de la température de traitement. Par traitement au chaud, le champ magnétique étant appliqué au long du laminage, la surface des domains transversales diminue et l'aimantation s'accomplit par un léger mouvement des parois des domains antiparallèles, ce que conduit à des faibles pertes par hystérésis [3].

3.2. Influence du temps de traitement et du champ magnétique appliqué

Dans le tableau 2 nous donnons les résultats concernants l'influence du temps de traitement et du champ magnétique. On voit qu'aux champs plus intenses appliqués pendant le traitement on obtient des échantillons

Tableau 2

l'influence du temps de traitement et du champ magnétique appliqué sur les propriétés magnétiques

Durée h	Température de traitement °K	Champ magnétique appliqué A/m	$\frac{H_c}{H_{c_0}}$	$\frac{B_r}{B_{r_0}}$	$\frac{B_{max}}{B_{max_0}}$	$\frac{P_{1T}}{P_{1T_0}}$	$\frac{P_{1,5T}}{P_{1,5T_0}}$
2	870	$1{,}2 \times 10^4$	1,05	23,64	12,30	0,21	0,09
2	870	$2{,}0 \times 10^4$	0,92	25,68	12,30	0,21	0,36
2	1 070	$2{,}0 \times 10^4$	0,66	23,18	12,07	0,11	0,36
2	1 070	$4{,}0 \times 10^4$	0,63	26,22	12,55	0,06	0,27
1	870	$1{,}2 \times 10^4$	0,97	26,22	12,55	0,19	0,11
2	870	$1{,}2 \times 10^4$	1,05	23,64	12,30	0,21	0.09
1	870	$2{,}0 \times 10^4$	1,00	26,36	12,30	0,18	0,17
2	870	$2{,}0 \times 10^4$	0,92	25,68	12,30	0,21	0,36

à B_r plus grand et à moindre H_c. On pourrait expliquer cette ci par une moindre dispersion des axes [001] de la direction du laminage.

Les résultats sur Fe-Si sont semblables aux résultats des études [5], [6] qui ont montré, sur des échantillons mono et policristalins du Fe, que H_c diminue avec le temps de traitement.

Toujours dans le tableau 2 on voit que l'augmentation du champ et du temps de traitement conduit à la diminution des pertes par hystérésis, mais les mesurements ne sont pas concludents en ce que concerne l'influence du temps de traitement.

4. Conclusions

1. Les caractéristiques d'hystérésis de l'alliage Fe-Si dépendent des conditions de traitement.

2. Par l'application d'un champ magnétique de saturation durant le traitement thermique, les pertes par hystérésis diminuent plus que par un traitement en absence du champ.

3. Par traitement thérmomagnétique on arrive à augmenter l'induction rémanente et à diminuer le champ coércitif à cause de l'orientation de domains suivant la direction du laminage.

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Centre de Recherches Techniques et Physiques, Jassy
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CARACTERISTICILE DE HISTEREZIS ALE ALIAJULUI Fe-3,27% Si LAMINAT ŞI TRATAT TERMOMAGNETIC

(Rezumat)

Se studiază influenţa tratamentului termomagnetic asupra caracteristicilor de histerezis ale aliajului Fe-3,27% Si laminat, în funcţie de temperatura de tratament, timpul de tratare şi intensitatea cîmpului magnetic aplicat în timpul tratamentului.

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D.C.551.515

ON THE POSSIBILITY TO DETERMINE THE CAUSE OF THE SC EVENTS BY USING THE CRITERION OF THE RISE SPEED $\Delta H/\Delta t$ OF THE HORIZONTAL COMPONENT OF THE GEOMAGNETIC FIELD

BY

ELENA BRÎNDUŞA BRADU and AL. MOLDOVANU

1. Introduction

From our previous works [1], [2], it follows that most of the storm sudden commencements, SC, are accounted for by the sector boundaries and the distortions within magnetic sectors in the solar wind, of the type of the subsectors or filaments.

But there are, also, numerous cases where the association between the individual solar flares and the SC events is obvious and this is the only mechanism explaining the corresponding geomagnetic disturbance [3].

Over the last years, investigations were conducted to explain both the physical characteristics of the shock waves associated with the solar flares and with the sector boundaries, on the basis of the direct measurements in the interplanetary space, as well as the morphological peculiarities of the corresponding SC events.

Thus, Nishida [4] tried to introduce the criterion of the duration Δt of the geomagnetic impulse. He suggested that the impulses with their rise time $\Delta t \leqslant 2$ minutes should be assigned for to the shocks, and the impulses which last more than 2 minutes to some small amplitude waves or to the tangential discontinuities.

Burlaga and Ogilvie [5], reanalyzing the results of Nishida, came to the conclusion that the criterion of the rise time of the geomagnetic impulse is not edifying enough to specify the nature of the distortion in the solar wind giving rise to it.

Siscoe et al. [6] found a theoretical relation of proportionality between the magnitude ΔH of the sudden geomagnetic impulse (SI) and the corresponding change in the solar wind pressure, but the experimentally determined proportionality constant is only one half of the theoretical value.

Taylor [7] concluded that the „larger" SC events were caused by the possible shocks, while the small ones were due to the tangential discontinuities.

But the quoted authors do not make any specification about the causes of the so called „shocks" and „tangential discontinuities", considering them as such in accordance with the parameters measured in the solar wind.

In this work, we do not consider the detailed physical characteristics of the distortions setting of the SC events. In return, we are trying to check up the SC events if they present any morphological peculiarities depending on their probable cause. Further more, since Wilcox and Colburn [8] and Moldovanu and Bradu [2] showed that numerous SC events were associated with the sector boundaries, we admit as a working hypothesis that numerous sector boundaries are preceded or succeeded by shock waves, the interface itself having characteristics of a tangential discontinuity.

2. Results and Discussion

In this work, 141 SC events were analyzed, from October 1966 to October 1969, for the periods where the data about magnetic sectorial structure of the solar wind were available (Wilcox and Colburn [8] and Wilcox and Colburn [9]).

A thoroughful examination of the magnetograms pointed out that the SC events seeming to be clearly associated with the sector boundaries were different from those associated with the solar flares.

Having noticed that the rise speed $\Delta H/\Delta t$ of the horizontal component of the geomagnetic field during the SC events was higher for the events associated with flares than for the events associated with sector boundaries, we proposed ourselves to check up statistically if one might see a grouping of these events depending on the criterion of $\Delta H/\Delta t$ and on their most probable cause.

The SC events of the considered interval were divided into 4 classes, depending on their most probable causes, as it follows:

1) The SC events generated by solar flares are considered those events preceded by flares of the importance 2 or higher within an interval of three days and unassociated with sector boundaries (fig. 1, graph *A*).

2) The SC events associated with sector boundaries and preceded by important solar flares were included into a special class (fig. 1, graph *B*).

3) The SC events associated with sector boundaries and not preceded by important solar flares (fig. 1, graph *C*).

4) The SC events unassociated with sector boundaries and not preceded by important solar flares (with an unidentified cause) (fig. 1, graph *D*).

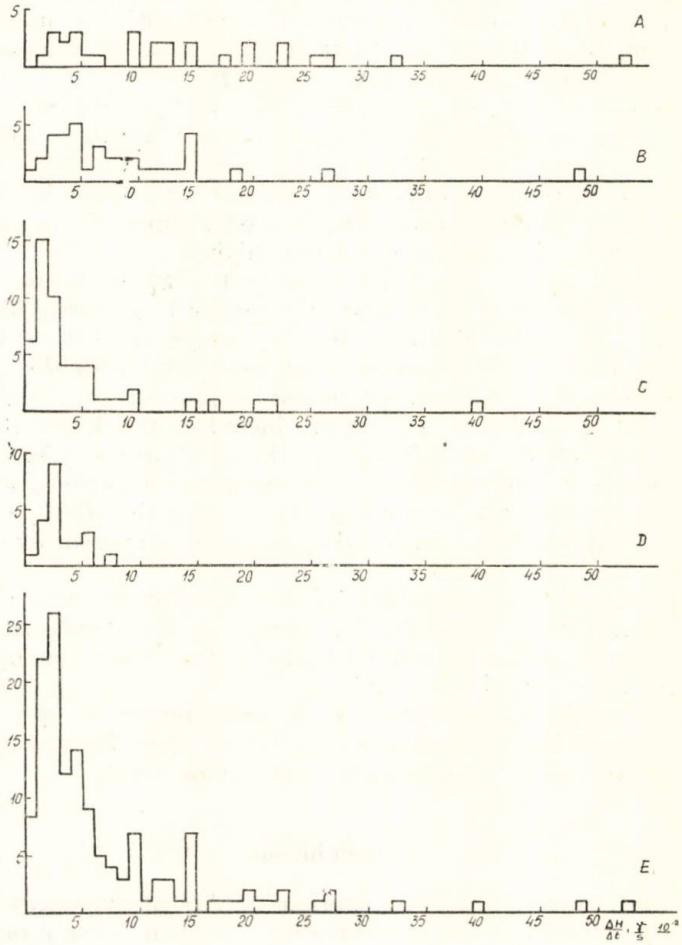

Fig. 1 — Statistical distribution (number of events) of SC events according to the criterion of $\Delta H/\Delta t$ and to their most probable cause: *A* — SC events associated with solar flares; *B* — SC events associated with solar flares and with sector boundaries; *C*—SC events associated with sector boundaries; *D*—SC events of an unknown cause; *E* — amount of all the SC events.

As a matter of fact, the grouping of the SC events according to these four classes is not absolutely certain and it might be valid only for a statistical treatment of the question.

In the fig. 1, the distribution of the investigated SC events is presented, in accordance with the four classes and the criterion of $\Delta H/\Delta t$. One can see that the introduction of this criterion renders evident a clear difference among the four classes of SC's. Thus, of the total number of cases, only 29 (i. e. 20,6 %) seem to be clearly associated with the individual solar flares. In return, 53 of the SC events (37,6 %) are obviously associated with the sector boundaries (fig. 1, graph *A* and *B*).

One can observe in fig. 1 graph *C*, that 81 % of the SC events associated with sector boundaries have their rise speed $\Delta H/\Delta t \leqslant 0{,}06$ gamma/sec.

With respect to the SC events associated with the solar flares, only 34 % of them have the rise speed $\Delta H/\Delta t \leqslant 0{,}06$ gama/sec, most of the SC in this class having their rise speed much higher.

It is worthy mentioning that, among the 37 SC events (graph *B*) associated both with solar flares and with sector boundaries, about a half (46 %) present a rise speed $\Delta H/\Delta t \leqslant 0{,}06$ gamma/sec and more than a half (54 %) have a distribution similar to those associated with the solar flares. This situation might be interpreted in two manners:

a) In this class, the SC events are included, which are produced, to an almost equal extent, by only one of the two possible causes.

b) These events emerge from some complex distortions, arising from the interaction between the sector boundaries and the shock waves given by the solar flares. Such an explanation was also proposed by Ness et al. [10].

Finally, from the examination of the distribution of the SC events of an undetermined cause (graph *D*), it follows that the events in this class have a distribution absolutely similar to that of the events associated with the sector boundaries (graph *C*).

Since, in addition, these events do not appear to be produced by solar flares, we consider them accounting for by some distortions connected with the subsectors and the filaments in the solar wind.

3. Conclusions

1. The introduction of the criterion of $\Delta H/\Delta t$ evidences a different distribution of the SC events, depending on their most probable cause. The SC events associated with the solar flares are steeper, while those associated with the sector boundaries are smoother. At the same time, it was statistically shown that the SC events associated with the solar flares are less numerous than those associated with the sector boundaries.

2. The determination of the rise speed $\Delta H/\Delta t$ of the SC events relies on the magnetograms at the observatory of Surlari, Romania, on a middle

latitude. Since the SC events show an effect of assimetry on the interplanetary scale, according to the longitude of the site, this effect should be corrected by using the data of the international network of observatories.

We think that performing this correction would not alter the obtained results; on the contrary, it would lead to a clearer distribution.

3. The criterion of the rise speed $\Delta H/\Delta t$ has not been used in literature so far. Our results show that this criterion offers additional information about the probable cause of the SC events. At the same time, the clear difference between the SC events depending on those two causes suggests that the corresponding shock waves have different characteristics and this fact presents a particular interest.

4. Of course, our results are valid only statistically and it is necessary to reexamine a number as large as possible of SC events in our statistics, from the standpoint of the parameters of the solar wind, as measured in the interplanetary space. This would, also, allow the elaboration of some new proper theoretical patterns.

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ASUPRA POSIBILITĂŢII DE DETERMINARE A CAUZEI EVENIMENTELOR SC DUPĂ CRITERIUL VITEZEI DE VARIAŢIE $\Delta H/\Delta t$ A COMPONENTEI ORIZONTALE A CÎMPULUI GEOMAGNETIC

(Rezumat)

Se discută posibilitatea determinării cauzelor evenimentelor SC după crititeriul vitezei de variaţie $\Delta H/\Delta t$ a componentei orizontale a cîmpului geomagnetic. Se remarcă statistic o diferenţă morfologică a evenimentelor SC. Cele datorate undelor de şoc generate de erupţiile solare cromosferice sînt mai abrupte şi au valorile $\Delta H/\Delta t$ mai mari, iar cele datorate trecerii peste Pămînt a limitelor sectoriale sînt mai netezite şi se grupează strîns spre valori mici ale lui $\Delta H/\Delta t$.

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D.C.535; 546.87

THE ACTION OF LIGHT ELECTROMAGNETIC WAVES ON THE THIN FILMS OF Bi [1])

BY

SOFIA MELINTE

1. Introduction

The thin films of Bi have constituted a study subject, especially a structural one, by F. Grob [1], Levinstein and Crane [2], Modeleine Gandais [3].

Madeleine Gandais has studied the structure of thin Bi films by the transmission and electronic difraction as well as by X-ray difraction. She employs the electronic diffraction by reflexion, and finds out that difraction in Bi — quite different to the X - ray diffraction that are produced in depth of films — takes place but on the surface.

Any study on the optic properties of these films has not yet been enterprised up to the present.

Having in view that Bi is a semi-metal and it has special properties as all the transition-elements, we have been persuated to enterprise a systematic study of Bi, of the optical point of view.

2. Films

The films have been obtained by the usual technic of thermic vaporization in vacuum, given by [4]. The pressure at which the films have deposed was of 1.210^{-6} torr.

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[1]) Presented at the National Symposium on „Radiation Interaction with Solids", Cluj, april 1970.

3. Determination of Bi Optical Constants

The optical constants R, R', T, A, A', (R being the reflexion factor of the film reported to air, R' — the reflexion factor of the film reported to the support, T — the transmission factor, A — absorbtion factor reported to air, A' — absorbtion factor reported to the support), have been determined by means of an optical method used in normal incidence by [4].

In the visible range one has worked for $\lambda = 500$, 589 and 700 mμ.

In the Fig. 1 the Bi films are seen to be less transparent for great wavelengths, and more transparent for small wavelenghts. The transmission also decreases with the film thicknes increasing.

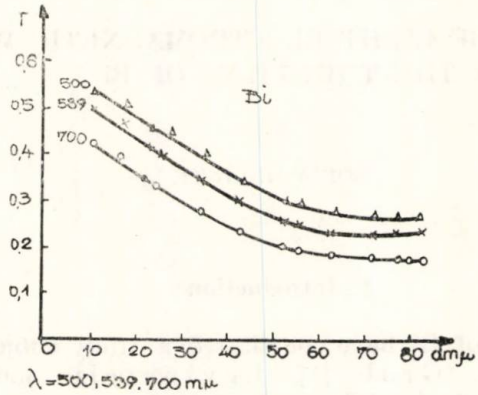

Fig. 1 — Variation of transmission factors in terms of the Bi film thickness for $\lambda = 500$, 589 and mμ.

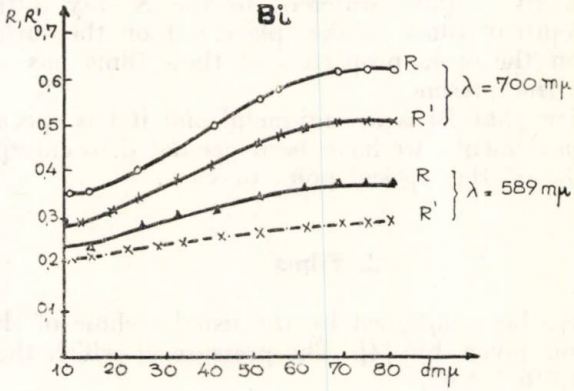

Fig. 2 — Variation of reflexion factors R and R' in terms of the Bi film thickness, for $\lambda = 589$ and 700 mμ.

In Fig. 2, the reflexion factors R and R' are seen to increase with the film thickness for both wavelenghs, but beyond a 60 mμ range they remain approximately constant. The reflexion also rises with the wavelengths growing. Since for $\lambda = 500$ mμ we have found a particular phenomenon, that is we have obtained antireflectant films during our experiences, we have separately represented the reflexion factors in terms of the specimen thickness.

The point for which some anti-reflectant mirrors are obtained may be found in Fig. 3 and 6, marked with (X).

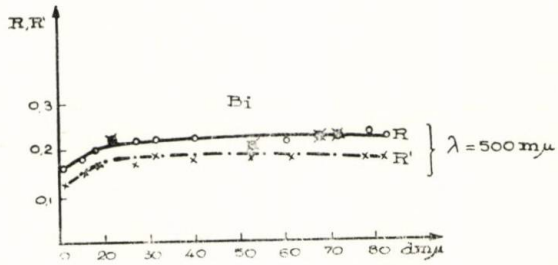

Fig. 3 — Variation of reflexion factors R and R' in terms of the film thickness of Bi for $\lambda = 500$ mμ.

4. Phenomenon Explanation

In order to clear up that problem we shall state further on the necessary conditions for obtaining antireflectant mirrors: in the first approximation it is sufficient for the amplitudes R and R' for both their sides (against air and the supports) to be equal in their modullus and to be opposite in phase, the multiple reflexions being neglected.

The conditions which the calculus leads to, are the following:

$$n^2 = n_0 n_2 , \qquad nd = \frac{\lambda}{4},$$

where: n_0 — refractions index of air, n_2 — refraction index of support, n — refraction index of the film and d — film thickness.

It is not easy to find a stuff able to fulfill both the conditions. Some more antireflectant mirrors of F_2Ca, have already been obtained as yet, but they are brittle and not to be applied in practice.

The Bi antireflectant film we have obtained is quite adherent to the support and has many possibilities to be applied in practice. The first and the most important application of that film is the control of the thin film thickness during the deposing, for $\lambda = 500$ mμ.

The efective energy absorbet in, to the Bi lamellas has been found by the relations :

$A = 1 - (R + T)$, when absorbtion is considered against air ;

$A' = 1 - (R' + T)$, when absorbtion is considered against the support.

Fig. 4, 5 and 6 show the Bi absorbtion when wavelength are 700, 589 and 500 mμ.

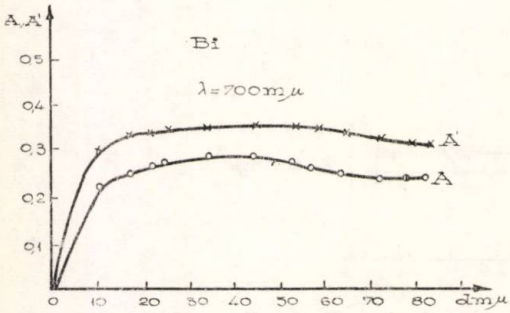

Fig. 4 — Variation of absorbtion factors A and A' for Bi, in terms of the film thickness, for $\lambda = 700$ mμ.

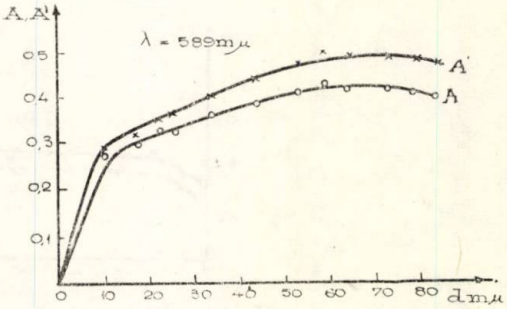

Fig. 5 — Variation of absorbtion factors A and A' for Bi, in terms of the film thickness, for $\lambda = 589$ mμ.

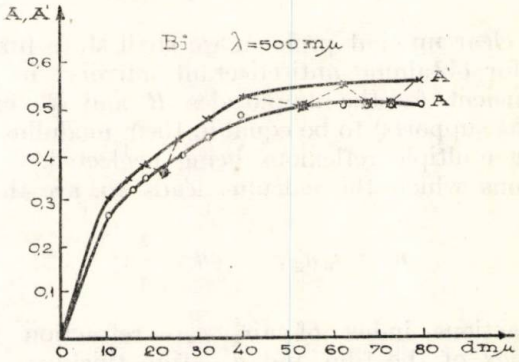

Fig. 6 — Variation of absorbtion factors A and A' for Bi, in terms of the film thickness, for $\lambda = 500$ mμ.

From the three diagrams one may note that the absorbtion power is increasing with the film thickness, as well as with the decreasing of wavelength, viz the frequency increasing. For instance, while for $\lambda = 700$ mμ, $d = 80$ mμ, $A = 25\%$; for $\lambda = 500$ mμ, and the same thickness, $A = 50\%$.

5. Conclusions

1. A powerful absorbtion phenomenon — that increases with the film thickness and the frequency increasing, that is the wavelength decreasing — is found into the films of Bi.

2. Some antireflectant films have been obtained, quite adherent to the support resistive, and being able to be applied in the thickness control of thin films, during the deposit, for $\lambda = 500$ mμ.

3. Some other antireflectant films of F_2Ca, but brittle and without any applicability in practice, are mentioned to have been obtained by other authors.

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ACŢIUNEA UNDELOR ELECTROMAGNETICE LUMINOASE, ASUPRA STRATURILOR SUBŢIRI DE Bi

(Rezumat)

Se studiază proprietăţile optice ale straturilor subţiri de Bi, depuse prin evaporare termică în vid. Grosimea straturilor este cuprinsă între 4,5 şi 80 mμ. Studiul are loc în domeniul vizibil. De remarcat este faptul că s-au obţinut oglinzi antireflectante simple, puternic aderente la suport, cu posibilităţi multiple de aplicare în practică, în special la controlul grosimii straturilor subţiri în timpul depunerii.

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Tomul XX (XXIV), Fasc. 3—4 1974

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MATEMATICĂ
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D. C. 517.43

O CLASĂ DE OPERATORI ŞI ECUAŢII CORESPUNZĂTOARE

DE

AL. CLIMESCU, M. GONTINEAC şi D. MOTREANU

Sub acest titlu sau echivalentul lui în alte limbi, colectivul îşi propune să studieze o clasă de operatori, liniari şi eventual neliniari, precum şi problema inversării lor.

Cei mai simpli din ei apar sub forma unor operatori liniari definiţi pe spaţiul amorf al şirurilor numerice indexate prin multiplii unui acelaşi h_p raţional (h_p putînd lua diverse valori, pentru regularitate se ia $h_p = 1/p$!). Convenţional aceşti operatori se vor numi operatori de derivare, un operator de derivare la dreapta de ordinul 1 definindu-se prin regula :

$$a'_n = \alpha_0 a_n + \alpha_1 a_{n+1} + \cdots + \alpha_p a_{n+p}, \tag{1}$$

unde α_i sînt numere satisfăcînd la relaţiile scrise pentru simplitate numai pentru $h_p = 1$:

$$\begin{aligned} \alpha_0 + \alpha_1 + \cdots + \alpha_p &= 0, \\ \alpha_1 + 2\alpha_2 + \cdots + p\alpha_p &= 1 \end{aligned} \tag{2}$$

şi similar pentru operatorii la stînga sau generali.

Cu ajutorul acestor operatori se pot defini ecuaţii diferenţiale pe numere raţionale, problemă care nu pare să fi fost sistematic atacată pînă în prezent, dar care are conexiuni atît cu ecuaţiile diferenţiale ordinare cît şi cu ecuaţiile cu diferenţe.

În ceea ce priveşte un cadru mai general pentru această problemă, în prezent colectivul are în vedere următoarele formulări şi probleme :

I. Definirea într-un mod convenabil a elementelor „raţionale" într-un spaţiu abstract amorf sau divers structurat.

II. Studiul problemelor corespunzătoare într-un spaţiu liniar topologic sau o familie de spaţii, inclusiv trecerea la limită inductivă, respectiv proiectivă a familiei de spaţii, cînd existenţa lor este asigurată.

III. Legătura cu geometria afină tratată laticial.

IV. Eventual introducerea unor structuri noi algebrico-topologice, mai flexibile în raport cu problemele puse.

V. Posibile aplicaţii concrete ale ecuaţiilor diferenţiale pe numere raţionale în unele probleme cu substrat tehnic.

Membrii actuali ai colectivului şi în viitor şi alţii eventual, vor ataca în chip individual unul sau altul din aspectele problemei, rămînînd solidar răspunzători de problema în ansamblu.

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A CLASS OF LINEAR OPERATORS AND CORRESPONDING EQUATIONS

(Summary)

The problem of differential equations on rationals is not yet developed, but seems to be an interesting one. The team of authors intend to attack the problem in the „concrete" form (i. e. numerical rationals) and also the more abstract aspects of it, including some extensions of the concept of rationals in abstract spaces and the thorough study of them in the case of linear topological spaces and their inductive or projective limits.

This paper, the first of a projected serial, has a programmatic character.

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NOTE SUR QUELQUES CONSÉQUENCES DE LA THÉORIE DES DÉVELOPPEMENTS ORTHONORMALES

PAR

VIOREL MURGESCU

Le but de cette Note est d'établir, dans des conditions assez générales, quelques conséquences presque immédiates de la théorie des développements orthonormales, lesquelles, dans des conditions particulières, ont été démontrées dans [1].

On traite sur ces questions parce qu'on croit que les inégalités qu'on se propose d'établir pourraient être utiles à l'évaluation des moyennes dans des calculs spéciaux de statistique.

Théorème 1. *Soit $\{a_k\}$ un ensemble fini de nombres réels et soit encore $\{\varphi_k\}$, $(k = 1, 2, \ldots, n)$, un ensemble fini de fonctions linéairement indépendantes, bornées sur $E \subset R^p$ et appartenant à un espace $L_2^G(E)$, où E est un ensemble de $\mathcal{L}$-mesure finie et $G(P)$, $(P \in E)$, est une fonction de mesure (f. positive, additive et normale), quelconque.*

Alors, il existe au moins une fonction bornée $f(P) \in L_2^G(E)$ et un système orthonormal $\{\psi_k(P)\}$, appartenant à l'espace linéaire engendré par $\{\varphi_k\}$, de sorte que les nombres a_k sont les coefficients de Fourier de la fonction $f(P)$, $(P \in E)$, par rapport au système $\{\psi_k\}$, c'est à dire

$$a_k = \mathcal{L}\int_E f(P)\psi_k(P)\, G(\mathrm{d}E), \quad (k = 1, 2, \ldots, n), \tag{1}$$

et on a

$$\frac{1}{n}\sum_{k=1}^{n} |a_k| \leqslant \frac{1}{\sqrt{n}}[G(E)]^{1/2} \sup_{P\in E} |f(P)|. \tag{2}$$

Démonstration. Vu que les fonctions φ_k sont linéairement indépendantes sur E, en vértu du théorème de E. Schmidt, on peut construire un système $\{\psi_k\}$ de fonctions orthonormales sur E, de la forme

$$\psi_k = \sum_{j=1}^{n} \lambda_{kj}\varphi_j, \quad (k = 1, 2, \ldots, n),$$

où les nombres λ_{kj} sont bien déterminés par les conditions d'orthonormalité,

$$\mathcal{L}\int_E \psi_k \psi_i \, G(\mathrm{d}E) = \delta_{ki}, \quad (k, i = 1, 2, \ldots, n).$$

Alors, en prenant

$$f(P) = \sum_{k=1}^{n} a_k \psi_k(P), \quad (P \in E),$$

on constate que les relations (1) sont satisfaites et, compte tenu d'abord de l'inégalité de Cauchy - Schwarz et puis de l'inégalité de Bessel, on a

$$\sum_{k=1}^{n} |a_k| = \sum_{k=1}^{n} |a_k| \cdot 1 \leqslant \left(\sum_{k=1}^{n} a_k^2\right)^{1/2} \left(\sum_{k=1}^{n} 1^2\right)^{1/2} = n^{1/2} \left(\sum_{k=1}^{1} a_k^2\right)^{1/2} \leqslant$$

$$\leqslant n^{1/2} \left[\mathcal{L}\int_E f^2(P) G(\mathrm{d}E)\right]^{1/2} \leqslant n^{1/2} [G(E)]^{1/2} \sup_{P \in E} |f(P)|.$$

En divisant partout par n, on obtient justement l'inégalité (2).

Son importance réside dans le fait qu'elle fournit une limitation supérieure de la moyenne des valeurs absolues des nombres a_k à l'aide d'un ensemble fini de fonctions du $L_2^G(E)$, qui leur est attaché par une voie quelconque.

Parfois on peut raffiner l'inégalité (2) en raisonant d'une manière un peu différente, comme dans la suite.

Théorème 2. *Dans les conditions du théorème* 1, *si l'ensemble fini* $\{\varphi_k\}$ *peut être inclus dans un ensemble fermé* $\{\bar{\varphi}_k\} \subset L_2^G(E)$, *alors il existe un système orthonormal* $\{\psi_k\} \subset L_2^G(E)$, *engendré par* $\{\bar{\varphi}_k\}$, *et encore, il existe une fonction* $g(P) \in L_2^G(E)$, *de sorte que les nombres* a_k *sont les premiers coefficients de Fourier de la fonction* $g(P)$ *par rapport au système* $\{\psi_k\}$ *et on a*

$$\frac{1}{n} \sum_{k=1}^{n} |a_k| \leqslant \frac{1}{\sqrt{n}} \|g\|_E^G, \tag{3}$$

où

$$\|g\|_E^G = \left[\mathcal{L}\int_E g^2(P)\, G(\mathrm{d}E)\right]^{1/2}. \tag{4}$$

Démonstration. L'ensemble fermé $\{\bar{\varphi}_k\}$, où $\bar{\varphi}_j = \varphi_j$, $(j = 1, 2, \ldots, n)$, peut engendrer, en vertu du théorème de E. Schmidt, un système orthonormal et fermé $\{\psi_k\}$ sur E. D'autre part, on peut compléter l'ensemble fini $\{a_k\}$, $(k = 1, 2, \ldots, n)$, par une suite arbitraire

$$a_{n+1}, a_{n+2}, \ldots, a_{n+p}, \ldots,$$

mais satisfaisant seulement la condition

$$\sum_{k=1}^{n} a_k^2 < +\infty.$$

Alors, en vertu du théorème de Riész-Fischer, il existe une fonction $g(P) \in L_2^G(E)$, $(P \in E)$, de sorte qu'on a

$$a_k = \mathcal{L} \int_E g(P)\, \psi_k(P)\, G(\mathrm{d}E), \qquad (k = 1, 2, \ldots).$$

En faisant de nouveau les majorations utilisées dans la démonstration du théorème précédent et encore, en faisant la notation (4), on arrive à l'inégalité (3).

Vu que la fonction $g(P)$ satisfait la condition de fermeture de Parseval, il est possible quelquefois qu'on a

$$\left| \frac{1}{n} \sum_{k=1}^{n} |a_k| - \frac{1}{\sqrt{n}} \| g \|_E^G \right| \leqslant \left| \frac{1}{n} \sum_{k=1}^{n} |a_k| - \frac{1}{\sqrt{n}} [G(E)]^{1/2} \sup_{P \in E} |f| \right|.$$

Observation. En particulier, pour $E = [a, b] \subset R$, $\{\varphi_k\}$, $(k = 1, 2, \ldots, n)$, étant un système orthonormal et continu sur $[a, b]$ et pour $G(\mathrm{d}E) = \omega(x)\, \mathrm{d}x$, où $\omega(x) \geqslant 0$, on obtient le théorème établi dans [1], lequel, partiellement, constitue la résolution de l'exercice 31 ([2], p. 125).

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NOTĂ ASUPRA UNOR CONSECINŢE ALE TEORIEI DEZVOLTĂRILOR ORTONORMALE

(Rezumat)

Se stabilesc două inegalităţi ((2) şi (3)) care precizează o limitare superioară a mediei aritmetice a valorilor absolute a n numere reale oarecare, acestea apărînd ca o consecinţă a unor dezvoltări ortonormale în spaţii $L_2^G(E)$.

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SOME REMARKS CONCERNING THE COVERING SPACES OF THE MANIFOLDS WITH BOUNDARY

BY

I. POP and FULGA POP

In this paper there are given some remarks concerning the relations between the covering spaces of the manifolds N, $N - \partial N$ and ∂N, where N is a differentiable manifold having ∂N as boundary.

1. Proposition 1. *Let N be a connected manifold, with boundary, having the dimension $n \geqslant 1$, and let $q : D \to N - \partial N$ be a differentiable covering map.*

Then, there exists a differentiable covering map $p : M \to N$ such that $p/M - \partial M : M - \partial M \to N - \partial N$ is equivalent with the covering map q.

Proof. (The proof of this proposition is similar to the proof of the Proposition 2.2 Ch. X of [1]).

Since N is a connected manifold, it follows that $N - \partial N$ is also connected and, therefore, q has a constant covering index. Let be $x \in \partial N$. There exists V an open neighbourhood of x and $\varphi : V \to R^{n-1} \times [0, \infty)$ a homeomorphism such that $\varphi(V \cap \partial N) = R^{n-1}$. Let be the diagram

$$\begin{tikzcd} D \arrow[r, "q"] & N - \partial N \\ & V - (V \cap \partial N) \arrow[u, "j"'] \end{tikzcd}$$

The inclusion map j induces a covering map $q_V : D_V \to V - (V \cap \partial N)$. Because V is homeomorphic with $R^{n-1} \times [0, \infty)$ and since $V \cap \partial N$ is homeomorphic with R^{n-1}, it follows that $V - (V \cap \partial N)$ is homeomorphic with $R^{n-1} \times [0, \infty) - R^{n-1}$ which is a connected locally path connected and 1-connected space. Therefore the covering map q_V is equivalent with a trivial covering map. Let F be a discret space such that q_V is equivalent with the covering map $p_{pr_1} : [V - (V \cap \partial N)] \times F \to V - (V \cap \partial N)$. Then, we can consider the covering map $p_V = p_{pr_1} : V \times F \to V$. It is obvious that the inclusion $j_V : V - (V \cap \partial N) \to V$ induces, with respect to p_V, the previous covering map. Therefore the inclusion j induces, with respect to the covering map q, a covering map which is equivalent with that who is induced by the inclusion j_V with respect to the covering map p_V.

Let $f_V : [(V - (V \cap \partial N))] \times F \to D_V$ be the homeomorphism which gives the equivalence of the previous covering maps. We have the commutative diagram :

$$\begin{tikzcd}
D \arrow[r, "q"] & N - \partial N \\
D_V \arrow[u, "k"] \arrow[r, "q_V"] & V - (V \cap \partial N) \arrow[u, "j"'] \\
{[V - (V \cap \partial N)] \times F} \arrow[u, "f"] \arrow[d] \arrow[r] & V \arrow[d] \\
V \times F \arrow[r, "p_V"'] & V
\end{tikzcd}$$

where k is the inclusion $q^{-1}(V - (V \cap \partial N)) \to D$. Then, we can consider the adjunction spaces $(V \times F)\ V \vee_{kf} D$ and $V \vee_j V\ (N - \partial N)$ and the map P_V: $(V \times F) \vee_{kf} D \to V \vee_j (N - \partial N)$ induced by the map $p_V \vee q$ such that there exists the commutative diagram

$$\begin{tikzcd}
(V \times F) \times D \arrow[r, "p_V \vee q"] \arrow[d, "\pi"'] & V \times (N - \partial N) \arrow[d, "\pi'"] \\
(V \times F) \vee_{kf} D \arrow[r, "P"] & V \vee_j (N - \partial N)
\end{tikzcd}$$

where π and π' are canonical projections.

Because the space $V \vee (N - \partial N)$ is locally path connected and his connected components are V and $N - \partial N$ it follows that $p_V \vee q$ is a covering map. Moreover, since the attaching maps kf and j are injective, it follows that π and π' are local homeomorphism, therefore P is also a covering map.

It is obvious that P induces over $N - \partial N$ a covering map equivalent to q. Now, we consider a system $(V_i)_{i \in I}$ such that $(V_i \cap \partial N)_{i \in I}$ is a covering of ∂N and such that there exist the trivial covering maps $p_i : E_i \to V_i$ which induce over $V_i - (V_i \cap \partial N)$ the covering maps which are equivalent with those induced by q.

Let be $V_i \cap V_j \neq \Phi$. Then, the covering maps $p_i / p_i^{-1}(V_i \cap V_j)$ and $p_j / p_j^{-1}(V_i \cap V_j)$ have the same covering index because either is equivalent to $q^{-1}(V_i \cap V_j - ((V_i \cap V_j) \cap N))$. Using the previous construction we can consider the adjunction space at D of the spaces E_i by the identification of the spaces $p_i^{-1}(V_i - (V_i \cap \partial N))$ and $q^{-1}(V_i - (V_i - (V_i \cap \partial N))$. Also, we can consider the adjunction space at $N - \partial N$ of the spaces V_i by the attaching maps $V_i \supset V_i - V_i \cap \partial N \to N - \partial N$.

It is obvious that this last space coincides with N and we obtain a covering map $p : M = (\bigvee_i E_i) \vee D \to N$ which induces over D a covering map which is equivalent to q.

Corollary 1. *Let $p : M \to N$ be a differentiable map with M and N two manifolds with boundary. If the map $p/M - \partial M : M - \partial M \to N - \partial N$ is a covering map, then p is also a covering map.*

Proof. From the Proposition 1, there exists a covering map $p' : M' \to N$ such that $M' - \partial M'$ is diffeomorphic with $M - \partial M$. This diffeomorphism can be extended at a diffeomorphism $\psi : M' \to M$ such that $p\psi = p'$.

Corollary 2. *If $q : D \to N - \partial N$ is a differentiable covering map with N a connected manifold, then there exists a covering map $q' : D' \to \partial N$ such that D' is the boundary of certain manifold of the same dimension as N.*

Proposition 2. *If N is a connected manifold and $q_i : D_i \to N - \partial N$, $i = 1, 2$, are two equivalent covering maps, then the covering maps $p_i : M_i \to N$ induced by q_1 and q_2 respectivelly, proceeding from the Proposition 1, are equivalent.*

Proof. Let be $x_i \in M_i - \partial M_i$, $i = 1, 2$, such that $p_1(x_1) = p_2(x_2)$. Since $p_i/M_i - \partial M_i$ is equivalent to q_i, it follows that for $y_i \in D_i$ such that $p_i(x_i) = q_i(y_i)$, the groups $p_{i\#}\,\pi(M_i - \partial M_i, x_i)$ and $q_{i\#}\,\pi(D_i, y_i)$ are conjugated in the group $\pi(N - \partial N, p_i(x_i))$. We obtain that the groups $p_{1\#}\,\pi(M_1 - \partial M_1, x_1)$ and $p_{2\#}\,\pi(M_2 - \partial M_2, x_2)$ are conjugated in the group $\pi(N - \partial N, p_1(x_1))$. But $p_{1\#}\,\pi(M_1, x_1) = p_{1\#}\,\pi(M_1 - \partial M_1, x_1)$ and $p_{2\#}\,\pi(M_2, x_2) = p_{2\#}\,\pi(M_2 - \partial M_2, x_2)$. It follows that the groups $p_{1\#}\,\pi(M_1, x_1)$ and $p_{2\#}\,\pi(M_2, x_2)$ are conjugated in the group $\pi(N, p_1(x_1))$ and therefore p_1 and p_2 are equivalent.

Corollary 3. *There exists a one-to-one corespondance between the classes of equivalence of the covering space of the connected manifold N and those of the manifold $N - \partial N$.*

Corollary 4. *If N is a connected manifold and $q : D \to N - \partial N$ is a regular covering map, then the covering map $p : M \to N$ obtaining by the Proposition 1 is also regular.*

One proves that the group $p_{\#}\,\pi(M, x)$ is a normal subgroup of the group $\pi(N, q(y))$, for $x \in M - \partial M$ and $y \in D$ such that $p(x) = q(y)$.

Corollary 5. *If N is a connected manifold and $q : D \to N - \partial N$ is a universal covering map, then the covering map $p : M \to N$ obtaining by the Proposition 1 is also universal.*

Proposition 3. *A differentiable covering map $p : M \to N$, with M a connected manifold, is a regular covering map if and only if the map $p/M - \partial M : M - \partial M \to N - \partial N$ is a regular covering map.*

Proof. If $p/M - \partial M$ is a regular covering map, it follows, in view of the Corollary 4, that the covering map $q : M' \to N$ given by the Proposition 1 is regular. But q is equivalent to p and therefore p is regular.

Conversely, if p is a regular covering map, there exists $x \in M$ such that $p_{\#}\,\pi(M, x)$ is a normal subgroup of the group $\pi(N, p(x))$. Moreover, since M is a path connected space, there exists $x_1 \in M - \partial M$ such that $p_{\#}\,\pi(M, x) \simeq p_{\#}\pi(M, x_1) = p_{\#}\pi(M - \partial M, x_1)$. It follows that the group $\pi(N - \partial N, p(x_1))$ is isomorphic with the group $\pi(N, p(x_1))$.

Proposition 4. *Let $p : M \to N$ be a differentiable covering map, with M a connected manifold. Then, p is an universal covering map if and only if the map $p/M - \partial M : M - \partial M \to N - \partial N$ is an universal covering map.*

P r o o f. If M is a connected manifold, then $M - \partial M$ is also connected. If $p/M - \partial M$ is an universal covering map, then the covering map $p'M' \to \to N$ given by the Proposition 1 is also universal (Corollary 5) and p' is equivalent to p (Proposition 2).

Conversely, let be p universal and let $q' : D' \to N - \partial N$ be a covering map with D' a connected manifold. According to the Proposition 1, there exists $q : M' \to N$ a covering map which induces q'. In view of the assumption, there exists $f : M \to M'$ such that $qf = p$. Composing the map $f/M - - \partial M$ with the equivalence between $M' - \partial M'$ and D' we obtain a morphism from $p/M - \partial M$ at q', which proves that $p/M - \partial M$ is an universal covering map.

2. P r o p o s i t i o n 5. *Let N be a manifold with connected boundary and let $q' : D' \to \partial N$ be a covering map. Then, there exists a submanifold N' of N such that $\partial N' = \partial N$ and a covering map $p' : M' \to N'$ such that the inclusion map $\partial N' \to N'$ induces a covering map equivalent with q'.*

P r o o f. Let V be an open neighbourhood in N and φ: $V \to R^{n-1} \times \times [0, \infty)$ a homeomorphism such that $\varphi(V \cap \partial N) = R^{n-1}$. The inclusion map $j : V \cap \partial N \to \partial N$ induces a covering map $q_V : A_V \to V \cap \partial N$. Since $V \cap \partial N$ is homeomorphic with R^{n-1}, it follows that q_V is equivalent with a trivial covering map

Then, there exists $p_V : E_V \to V$ a trivial covering map such that it induces over $V \cap \partial N$ a covering map equivalent to q_V.

Now, we consider the covering maps $p_i : E_i \to V_i$, $i \in I$, such that $\{V_i \cap \partial N\}_{i \in I}$ is a covering for ∂N. Using the attaching maps $V_i \supset V_i \cap \partial N \to N$ we consider the adjunction space of $(V_i)_{i \in I}$ at ∂N and, identifying $p_i^{-1}(V_i \cap \partial N)$ with $q^{-1}(V_i \cap \partial N)$ we obtain the adjunction space of $(E_i)_{i \in I}$ at D'. The first space is a manifold N' such that $\partial N' = \partial N$ and it has the second as a covering space.

It is obvious that the restriction over ∂N of this covering map is equivalent with q'.

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UNELE OBSERVAŢII ASUPRA SPAŢIILOR DE ACOPERIRE ALE VARIETĂŢILOR CU BORD

(Rezumat)

Se arată că dacă N este o varietate conexă cu bord atunci există o corespondenţă biunivocă între clasele de echivalenţă ale spaţiilor de acoperire ale varietăţilor N şi $N - \partial N$. Prin această corespondenţă proprietatea unei acoperiri de a fi regulată sau universală se păstrează. Se cercetează de asemenea posibilitatea extinderii unei acoperiri a bordului ∂N de la o acoperire a unei vecinătăţi a acestuia.

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A NOTE ON THE SUMM OF TWO m-DISSIPATIVE MAPPINGS

BY

ADRIAN CORDUNEANU

Let X be a real Banach space and let X^* its dual space. The duality mapping $F: X \to X^*$ is given by the relation

$$Fx = \{x^* \in X^*;\ (x, x^*) = \|x\|^2 = \|x^*\|^2\}, \tag{1}$$

where by $(\ ,\ )$ we have denoted the pairing between X and X^*.

The mapping (graph) $A \subset X \times X$ is called dissipative if for every $[x_i, y_i] \in A$, $i = 1, 2$, there exists an element $x^* \in F(x_1 - x_2)$ such that $(y_1 - y_2, x^*) \leqslant 0$. The dissipative mapping A is called m-dissipative, if for every $\lambda > 0$ it follows $R(\lambda - A) = X$. We have wrote $R(\lambda - A)$ instead of $R(\lambda I - A)$, where I is the identity operator in X.

Are also well-known the folowing notations

$$J_n = (1 - n^{-1} A)^{-1}, \quad A_n = n(J_n - I). \tag{2}$$

The most remarkable properties of m-dissipative mappings are collected in [1, pp. 89—98].

Definition 1. The Banach space X is said to be uniformly convex, if for every $0 < \varepsilon \leqslant 2$ there exists a number $\delta(\varepsilon) > 0$ such that from $\|x\| = \|y\| = 1$, $\|x - y\| \geqslant \varepsilon$ it follows $\|x + y\| \leqslant 2(1 - \delta(\varepsilon))$.

Definition 2. The Banach space X is said to be locally uniformly convex, if for every x and ε with $\|x\| = 1$ and $0 < \varepsilon \leqslant 2$, there exists a number $\delta(\varepsilon, x) > 0$ such that from $\|y\| = 1$ and $\|x - y\| \geqslant \varepsilon$ it follows $\|x + y\| \leqslant 2(1 - \delta(\varepsilon, x))$.

We also shall employ the following result [2, p. 61].

Lemma. *Let X be a locally uniformly convex Banach space. Then, from the relations*

$$x_n \rightharpoonup x \quad \textit{and} \quad \|x_n\| \to \|x\| \tag{3}$$

it follows $x_n \to x$.

Here, the arrow $\rightharpoonup$ denotes the weak convergence in X.

Definition 3. A function $B: D(B) \subset X \to X$ is called locally bounded if for every $x \in D(B)$ there exists a neighbourhood $V(x, \rho(x))$ such that B is bounded on $V(x, \rho(x)) \cap D(B)$.

We shall prove the following

Theorem. *Let X be a locally uniformly convex Banach space with uniformly convex dual X^*. Assume that A and B are m-dissipative mappings in $X \times X$ and that*

(a) $D(A) \cap D(B) \neq \emptyset$.

(b) *$D(B)$ is closed, B is single valued and locally bounded.*

Then $A + B$ is m-dissipative.

Proof. Because X^* is uniformly convex, the duality map $F: X \to X^*$ is single valued and clearly follows that $A+B$ is dissipative on $D(A) \cap D(B)$

On the other hands, we can prove that for every $\lambda > 0$, $R(\lambda - A - B)$ is a closed set. Indeed, let $y_n \in R(\lambda - A - B)$ and let $y_n \to y$. If $x_n \in D(A) \cap D(B)$ is such that $\lambda x_n - Ax_n - Bx_n \ni y_n$, taking into account that A and B are dissipative, we have

$$\lambda \|x_m - x_n\| \leqslant \|y_m - y_n\| \tag{4}$$

for every natural numbers m and n. Thus (x_n) is a Cauchy sequence : $x_n \to x$. Because $D(B)$ is closed, clearly $x \in D(B)$. We can suppose that $x_n \in V(x, \rho(x))$ and consequently $\|Bx_n\| \leqslant M$ for every natural n, where $M > 0$ is a constant. Without any loss of generality, we can assume that (Bx_n) is a weak-convergent sequence in X. From the relations $x_n \to x$ and $Bx_n \rightharpoonup w$, we deduce that $w = Bx$. Denoting by z_n the element of Ax_n for which $\lambda x_n - z_n - Bx_n = y_n$, we also deduce that (z_n) is a weak-convergent sequence : $z_n \rightharpoonup z$ and that $z \in Ax$. Passing to the limit in the above equation, we conclude that $\lambda x - Ax - Bx \in y$, i. e. $y \in R(\lambda - A - B)$. Thus $R(\lambda - A - B)$ is a closed set in X, for every $\lambda > 0$.

Now, we shall prove that for every $\lambda > 0$, $R(\lambda - A - B)$ is a open set in X. Let $y_0 \in R(\lambda - A - B)$ be a fixed element. There exists $x_0 \in D(A) \cap D(B)$ such that $\lambda x_0 - z_0 - Bx_0 = y_0$, where z_0 is an element of Ax_0. Let $y \in X$ be a fixed element and let $x_n \in D(A)$ the unique solution of the equation $\lambda x_n - Ax_n - B_n x_n \ni y$. From the relations

$$\lambda x_n - Ax_n - B_n x_n \ni y, \quad \lambda x_0 - Ax_0 - Bx_0 \ni y_0 \tag{5}$$

we get in a standard manner that

$$\lambda \|x_n - x_0\| \leqslant \|y - y_0\| + \|B_n x_0 - Bx_0\|, \quad (n = 1, 2, 3, \ldots). \tag{6}$$

Because X^* is uniformly convex and $(B_n x_0)$ is bounded, it follows $B_n x_0 \rightharpoonup Bx_0$. From $B_n x_0 \rightharpoonup Bx_0$ and $\limsup_{n \to \infty} \|B_n x_0\| \leqslant \|Bx_0\|$, applying the cited above lemma, we obtain $B_n x_0 \to Bx_0$. Thus, if $\|y - y_0\|$ is sufficiently small, taking into account the inequality (6) it follows that $x_n \in V(x_0, \rho(x_0))$ for $n \geqslant N_0$. Because

$$B_n x_n = BJ_n x_n = B(1 - n^{-1} B)^{-1} x_n, \quad (n = 1, 2, 3, \ldots), \tag{7}$$

and $J_n x_n$ also is contained in $V(x_0, \rho(x_0))$ for $n \geqslant N_1$, we conclude that $(B_n x_n)$ is a bounded sequence in X. From the inclusion

$$(8) \quad \lambda(x_m - x_n) \in (Ax_m - Ax_n) + (B_m x_m - B_n x_n), \quad (m, n = 1, 2, 3....)$$

we obtain

$$(9) \quad \begin{aligned} \lambda \|x_m - x_n\|^2 &\leqslant (B_m x_m - B_n x_n, F(x_m - x_n)) \leqslant \\ &\leqslant (B_m x_m - B_n x_n, F(x_m - x_n) - F(J_m x_m - J_n x_n)) \end{aligned}$$

for every natural numbers m and n. If $C > 0$ is a constant such that $\|Bx_n\| \leqslant \leqslant C$ for every natural n, we have

$$(10) \quad \|x_m - x_n - (J_m x_m - J_n x_n)\| \leqslant C\left(\frac{1}{m} + \frac{1}{n}\right), \quad (m, n=1, 2, 3 ,...)$$

and if we remember that F is uniformly continuous on every bounded set of X, we conclude from (9) that (x_n) is a Cauchy sequence: $x_n \to x$. Finally, from $J_n x_n \to x$ and $B_n x_n = BJ_n x_n \to Bx$ we obtain that $y \in R(\lambda - - A - B)$ if $\|y - y_0\|$ is sufficiently small, i. e. $R(\lambda - A - B)$ is a open set. It follows that for every $\lambda > 0$, $R(\lambda - A - B) = X$ and the proof is complete.

Remark 1. The conclusion of the proved above Theorem also holds, if instead of (a) and (b) we assume

(a′) $D(A) \cap D(B)$

is closed.

(b′) B is single valued and locally bounded.

Remark 2. A similar result was proved by the author in [3].

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NOTĂ DESPRE SUMA A DOUĂ APLICAŢII m-DISIPATIVE

(Rezumat)

X fiind un spaţiu Banach local uniform convex cu dualul uniform convex, se demonstrează că suma a două aplicaţii A şi B m-disipative în X este de asemenea m-disipativă, dacă $D(A) \cap D(B) \neq \varnothing$, $D(B)$ este închisă şi B este local mărginită. Aceeaşi concluzie se menţine dacă admitem că $D(A) \cap D(B)$ este închisă şi B local mărginită.

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TEOREME DE PUNCT FIX PENTRU UNELE CLASE DE MULTIFUNCŢII

DE

AL. SABADÎŞ şi GH. FLOREA

Fie X un spaţiu Banach.

Definiţia 1. Spaţiul X se numeşte strict convex dacă frontiera sferei unitate nu conţine segmente, adică $\forall x_1, x_2 \in S = \{x \in X \mid \|x\| = 1\}$, $x_1, \neq x_2$, are loc relaţia $\|(1-t)x_1 + tx_2\| < 1$, $\forall t \in (0, 1)$.

În [2], se arată că o pereche de spaţii Banach reflexive pot fi renormate echivalent strict convex.

Definiţia 2. Spunem că spaţiul X are o aplicaţie duală, dacă există o aplicaţie $\varphi : R^+ \to R^+$ continuă şi strict crescătoare şi o aplicaţie $F_\varphi : X \to X^*$ continuă de la topologia slabă a lui X la topologia slabă a lui X^*, astfel încît oricare ar fi $x \in X$ are loc $(F_\varphi(x))(x) = \|F_\varphi(x)\| \cdot \|x\| = \varphi(\|x\|) \cdot \|x\|$. Funcţia φ se numeşte funcţie pondere.

Definiţia 3. Spunem că pe spaţiul X, cu X^* strict convex, iar $F : X \to X^*$ o aplicaţie duală, am definit o multifuncţie $V \subset X \times \mathcal{P}(X)$ parţial acretivă, dacă pentru fiecare pereche $(x, y) \in X \times X$ cu $V(x) \neq \varnothing$ şi $V(y) \neq \varnothing$ şi pentru orice $z \in V(x)$, există $w \in V(y)$ (w depinde de x şi y) astfel încît:

$$(F_\varphi(x - y))(z - w) \geqslant 0.$$

Propoziţie. Fie X un spaţiu Banach reflexiv cu X^* strict convex şi cu proprietatea că, oricare ar fi funcţia pondere φ, aplicaţia duală corespunzătoare $F_\varphi : X^*$ este continuă de la topologia slabă a lui X la topologia slabă a lui X^*. Dacă multifuncţia $V \subset X \times \mathcal{P}(X)$ are proprietăţile:

1) este parţial acretivă;

2) pentru orice $x \in X \Rightarrow V(x) \neq \varnothing$; convexă şi compactă în X;

3) V este semicontinuă superior şi surjectivă, atunci orice mulţime mărginită, închisă şi convexă $M \subset X$ are imaginea $V(M) = \overline{V(M)}$.

Definiţia 4. Fie $A \subset X$, $B \subset X$ cu $A = \overline{A}$ şi $B = \overline{B}$ unde X este un spaţiu metric. Prin distanţa dintre A şi B înţelegem numărul $\rho_p(A, B) = \max\{\rho(A, B), \rho(B, A)\}$ unde $\rho(A, B) = \sup\limits_{x \in A} \{\inf\limits_{y \in B} \rho(x, y)\}$ şi $\rho(B, A) = \sup\limits_{y \in B} \inf\limits_{x \in A} \rho(x, y)\}$.

D e f i n i ţ i a 5. Multifuncţia $\Gamma \subset X \times \mathcal{P}(X)$ cu $\Gamma(x) \neq \varnothing$ şi $\Gamma(x) = \overline{\Gamma(x)}$, $\forall\, x \in X$, se numeşte neexpansivă dacă $\rho_p(\Gamma(x), \Gamma(y)) \leqslant \rho(x, y)$, $\forall\,(x, y) \in X \times X$.

D e f i n i ţ i a 6. O multifuncţie $\Gamma \subset X \times \mathcal{P}(X)$ cu $\Gamma(x) \neq \varnothing$ şi $\Gamma(x) = \overline{\Gamma(x)}$, $\forall\, x \in X$, se numeşte contractantă dacă există $\alpha \in [0, 1)$ astfel încît

$$\rho_p(\Gamma(x), \Gamma(y)) \leqslant \alpha\rho(x, y), \qquad \forall\,(x, y) \in X \times X.$$

T e o r e m ă. Fie (X, ρ) *un spaţiu metric complet şi multifuncţia* $\Gamma \subset X \times \mathcal{P}(X)$. *Dacă* Γ *este contractantă şi* $\Gamma(x)$ *este nevidă şi închisă pentru orice* $x \in X$ *atunci există* $x_0 \in X$ *astfel încît* $x_0 \in \Gamma(x_0)$.

D e m o n s t r a ţ i e. Urmăm principiul de demonstraţie a teoremei lui Banach, referitoare la contracţiile obişnuite. Construim un şir de iteraţii pornind de la un punct fixat $\gamma_0 \in X$, ales arbitrar şi de la o constantă $\alpha_0 \in (0, 1)$. Presupunem că oricare ar fi $\gamma_1 \in \Gamma(\gamma_0)$ există $\gamma_2 \in \Gamma(\gamma_1)$ cu proprietatea $\rho(\gamma_2, \gamma_1) \leqslant \alpha_0\rho(\gamma_1, \gamma_0)$. Apoi vom presupune că există $\gamma_3 \in \Gamma(\gamma_2)$ cu $\rho(\gamma_3, \gamma_2) \leqslant \alpha_0\rho(\gamma_2, \gamma_1) \leqslant \alpha_0^2\rho(\gamma_1, \gamma_0)$ ş. a. m. d. Vom obţine astfel o mulţime de puncte $\{\gamma_0, \gamma_1, \ldots, \gamma_{n-1}\}$ cu proprietatea că $\gamma_j \in \Gamma(\gamma_{j-1})$, $\forall\, j \in \overline{1, n-1}$ şi $\rho(\gamma_j, \gamma_{j-1}) \leqslant \alpha_0^{j-1}\rho(\gamma_1, \gamma_0)$. Aşadar $\gamma_{n-1} \in \Gamma(\gamma_{n-2})$ şi $\rho(\gamma_{n-1}, \gamma_{n-2}) \leqslant \alpha_0^{n-2}\rho(\gamma_1, \gamma_0)$.

Prin ipoteză avem $\rho_p(\Gamma(\gamma_{n-1}), \Gamma(\gamma_{n-2})) \leqslant \alpha_0\rho(\gamma_{n-1}, \gamma_{n-2})$. Dacă $\rho(\gamma_{n-1}, \gamma_{n-2}) = 0$ rezultă $\gamma_{n-2} = \gamma_{n-1} \in \Gamma(\gamma_{n-2})$ şi teorema este demonstrată, punctul căutat fiind γ_{n-2}. În caz contrar există $\gamma_n \in \Gamma(\gamma_{n-1})$ astfel încît $\rho(\gamma_n, \gamma_{n-1}) \leqslant \alpha_0\rho(\gamma_{n-1}, \gamma_{n-2})$. Pe de altă parte avem $\rho_p(\Gamma(\gamma_{n-1}), \Gamma(\gamma_{n-2})) \leqslant \alpha_0\rho(\gamma_{n-1}, \gamma_{n-2}) \leqslant \alpha_0^2\rho(\gamma_{n-2}, \gamma_{n-3}) \leqslant \ldots \leqslant \alpha_0\,\alpha_0^{n-2}\,\rho(\gamma_1, \gamma_0) = \alpha_0^{n-1}\,\rho(\gamma_1, \gamma_0)$.

Deci punctul γ_n construit astfel se bucură de aceeaşi proprietate cu $\gamma_1, \gamma_2, \ldots, \gamma_{n-1}$. Continuînd construcţia punctelor γ_n de aceeaşi manieră vom obţine şirul $\{\gamma_n\}$. Cum

$$0 < \alpha_0 < 1 \Rightarrow \rho(\gamma_n, \gamma_m) \leqslant \rho(\gamma_n, \gamma_{n-1}) + \rho(\gamma_{n-1}, \gamma_{n-2}) + \ldots +$$
$$+ \rho(\gamma_{m+1}, \gamma_m) \leqslant \alpha_0^{n-1}\,\rho(\gamma_1, \gamma_0) + \alpha_0^{n-2}\,\rho(\gamma_1, \gamma_0) + \ldots + \alpha_0^m\,\rho(\gamma_1, \gamma_0) =$$
$$= \rho(\gamma_1, \gamma_0)\,[\alpha_0^{n-1} + \alpha_0^{n-2} + \ldots + \alpha_0^m] = \rho(\gamma_1, \gamma_0)\,\alpha_0^m\,\frac{1 - \alpha_0^{n-m}}{1 - \alpha_0} \leqslant$$
$$\leqslant \frac{\alpha_0^m}{1 - \alpha_0}\,\rho(\gamma_1, \gamma_0).$$

Rezultă $\lim\limits_{n,\,m} \rho(\gamma_n, \gamma_m) = 0$ şi prin urmare şirul $\{\gamma_n\}$ este un şir Cauchy. Cum (X, ρ) a fost presupus complet, rezultă existenţa unui punct $\gamma^* \in X$ cu proprietatea $\lim\limits_n \gamma_n = \gamma^*$. Deci pentru $\varepsilon > 0$, există $N(\varepsilon)$ astfel încît $\rho_p(\Gamma(\gamma_n), \Gamma(\gamma^*)) \leqslant \alpha_0\rho(\gamma_n, \gamma^*) \leqslant \varepsilon$ pentru $\forall\, n \geqslant N(\varepsilon)$. Rezultă că $\Gamma(\gamma_n)$ se află într-o vecinătate de rază ε a lui $\Gamma(\gamma^*)$ notată $V_\varepsilon[\Gamma(\gamma^*)]$. Dar $\lim\limits_n \rho(\gamma_n, \gamma^*) = 0$. Rezultă că există $V_{\varepsilon'(\varepsilon)}(\gamma^*)$ care intersectează $\Gamma(\gamma^*)$, independent de n, adică $\gamma^* \in \overline{\Gamma(\gamma^*)} = \Gamma(\gamma^*)$.

Să studiem unicitatea lui γ^*. Fie $\gamma^* \neq \gamma^{**}$ cu proprietatea $\gamma^* \in \Gamma(\gamma^*)$ şi $\gamma^{**} \in \Gamma(\gamma^{**})$. Avem $\rho(\gamma^*, \gamma^{**}) \leqslant d(\Gamma(\gamma^*)) + \rho_p(\Gamma(\gamma^*), \Gamma(\gamma^{**})) + d(\Gamma(\gamma^{**}))$, unde cu $d(\Gamma(\gamma^*))$ şi $d(\Gamma(\gamma^{**}))$ am notat diametrul mulţimilor din paranteză. Se vede de aici că $\rho(\gamma^*, \gamma^{**}) = 0$ numai dacă $d(\Gamma(\gamma^*)) = d(\Gamma(\gamma^{**}))=0$ şi $\Gamma(\gamma^*) \cap \Gamma(\gamma^{**}) \neq 0$ adică, dacă Γ ar fi funcţie obişnuită. Se pot da exemple de multifuncţii semiunivoce care admit o infinitate de puncte fixe. Astfel, fie $X = \{(x, y) \in R^2 \mid x > 0, y > 0, x^2 + y^2 \leqslant a^2\}$. Definim multifuncţia $\Gamma \subset X \times \mathcal{P}(X)$ prin $\Gamma(x, y)) = \Gamma(M) = \overline{OM}$, unde O este originea reperului. Aici oricare ar fi $M \in X$ rezultă $M \in \Gamma(M)$, deci mulţimea punctelor fixe ale multifuncţiei Γ este echivalentă cu X.

În [3] se dă o precizare asupra mulţimii punctelor fixe pentru o clasă particulară de multifuncţii. Dăm un rezultat analog pentru multifuncţii neexpansive.

T e o r e m ă. *Fie X un spaţiu Banach reflexiv şi X^* strict convex, astfel încît pentru orice funcţie pondere φ există o aplicaţie duală $F_\varphi : X \to X^*$, continuă de la topologia slabă a lui X la topologia slabă a lui X^*. Dacă Γ este o multifuncţie $\Gamma \subset X \times \mathcal{P}(X)$, cu proprietăţile:*

1) $\Gamma(x) \neq \Phi$, *compactă şi convexă*, $\forall x \in X$;

2) Γ *este neexpansivă*;

3) *există o mulţime $M \subset X$ mărginită, închisă şi convexă cu proprietatea că $\Gamma_M \subset M \times \mathcal{P}(M)$, atunci există $x_0 \in M$ cu proprietatea $x_0 \in \Gamma(x_0)$.*

D e m o n s t r a ţ i e. Presupunem că s-a făcut o translaţie astfel încît $0 \in M$. Definim o multifuncţie $\Gamma_\alpha \subset M \times \mathcal{P}(M)$ prin $\Gamma_\alpha(x) = \alpha\Gamma(x)$ unde $\alpha \in (0, 1)$. Pentru orice $(x, y) \in M \times M$ avem $\rho_p(\Gamma(x), \Gamma_\alpha(y)) \leqslant \alpha\rho_p(\Gamma(x), \Gamma(y)) \leqslant \alpha\rho(x, y)$, deci Γ_α este o contracţie şi admite cel puţin un punct fix în M. Deoarece M este mărginită şi închisă şi X este reflexiv, ea este compactă prin şiruri. Prin urmare, putem găsi un şir numeric $\alpha_j \to 1$ şi $y_{\alpha_j} \xrightarrow{s} y_0 \in M$ şi $y_{\alpha_j} \in \Gamma_{\alpha_j}(y_{\alpha_j}) = \alpha_j\Gamma(y_{\alpha_j})$. Deci $z_j = \alpha_j^{-1} y_{\alpha_j} \in \Gamma(y_{\alpha_j})$ şi $\|z_j - y_{\alpha_j}\| \to 0$ cînd $\alpha_j \to 1$. Prin urmare $y_{\alpha_j} - z_j \in (I - \Gamma)(M)$. Cum $y_{\alpha_j} - z_j \xrightarrow{t} 0$ în X, rezultă că $0 \in (I - \Gamma)(M)$.

Notăm cu $V = I - \Gamma$, unde Γ este neexpansivă, prin ipoteză. Fie $(y, z) \in G(\Gamma)$ şi $(y, y - z) \in G(V)$ iar $x \in X$, arbitrar. Din compacitatea lui $V(x)$ rezultă că, există $w \in V(x)$ cu proprietatea $\rho(z, w) \leqslant \rho(y, x)$, deoarece Γ este neexpansivă.

Arătăm că V este parţial acretivă. Într-adevăr, cu $y - z \in \Gamma(y)$ şi $x - w \in V(x)$ avem:

$$((y - z) - (x - w), F_\varphi(y - x)) = ((y - x), F_\varphi(y - x)) - ((z - w),$$

$$F_\varphi(z - w)) = \|y - x\| \varphi(\|y - x\|) - \|z - w\| \varphi(\|z - w\|) \geqslant$$

$$\geqslant \|y - x\| \varphi(\|y - x\|) - \|y - x\| . \varphi(\|y - x\|) = 0.$$

Conform propoziţiei de mai sus $0 \in V(M) = (I - \Gamma)(M)$. Deci există $x_0 \in M$ astfel încît $0 \in V(x_0)$, de unde rezultă $x_0 \in \Gamma(x_0)$, şi teorema este demonstrată.

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DES THEORÈMES DE POINT FIXE POUR QUELQUES CLASSES D'APPLICATIONS MULTIVOQUES

(Résumé)

On démontre quelques theorèmes de point fixe pour des applications multivoques $\Gamma \subset X \times \mathcal{P}(X)$ avec $\Gamma(x) \neq \emptyset$ et $\Gamma(x) = \overline{\Gamma(x)}$, $\forall x \in X$, en utilisant les résultats de la théorie des applications univoques.

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O CORESPONDENŢĂ ÎNTRE VARIETĂŢI RIGLATE IN S_4 ŞI SISTEMUL DE ECUAŢII DIFERENŢIALE

DE

ŞTEFANIA RUSCIOR şi DRAGOŞ ŞTEFANCU

1. În această notă ne propunem să stabilim o corespondenţă prin hiperplane tangente parţial paralele între două varietăţi riglate V şi V^* din S_4 afin.

Considerăm o corespondenţă între generatoarele g şi g^* de pe varietăţile V şi V^*. Vom determina o corespondenţă între punctele generatoarelor g şi g^*, astfel ca în punctele corespunzătoare $M \in g$ şi $M^* \in g^*$, hiperplanele tangente H şi H^* duse respectiv la V şi V^* să fie parţial paralele, cu dimensiunea paralelismului $d_p = \frac{2}{3}$ [1], [2].

Fie varietăţile riglate V şi V^*, date respectiv prin ecuaţiile:

$$\bar{r} = \bar{\rho}(u) + v\bar{a}(u) + w\bar{\alpha}(u, v), \qquad \bar{r}^* = \bar{\rho}^*(u, v) + w\bar{\alpha}^*(u, v). \tag{1}$$

Parametrii u, v, w au fost aleşi astfel încît pentru $u = \text{const}$ şi $v = \text{const}$ să fie determinate generatoarele corespunzătoare g şi g^* de direcţie $\bar{\alpha}$ şi $\bar{\alpha}^*$ cu $\bar{\alpha} \neq \bar{\alpha}^*$. Hiperplanele tangente H şi H^* sînt parţial paralele cu $d_p = \frac{2}{3}$, dacă au o dreaptă improprie comună [1], [2].

Ecuaţiile acestor hiperplane sînt:

$$\begin{aligned} &(H) \qquad [\bar{R} - \bar{r}, \bar{r}_u, \bar{r}_v, \bar{r}_w] = 0, \\ &(H^*) \qquad [\bar{R} - \bar{r}^*, \bar{r}^*_u, \bar{r}^*_v, \bar{r}^*_w] = 0. \end{aligned} \tag{2}$$

Putem realiza condiţia ca H şi H^* să fie parţial paralele de tipul considerat, dacă considerăm că planul tangent T_0 dus suprafeţei directoare S, în punctul unde g intersectează suprafaţa S este semiparalel cu planul tangent T_0^*, dus în punctul unde g^* intersectează suprafaţa directoare S^* de pe V^* [2], [3]. Suprafeţele S şi S^* corespund lui $w = 0$ şi $w^* = 0$. Cele două plane tangente T_0 şi T_0^* au un punct comun impropriu I_0.

Condiţia ca planele tangente T_0 şi T_0^* să fie semiparalele este dată de relaţia:

$$[\bar{\rho}', \bar{a}, \bar{\rho}_u^*, \bar{\rho}_v^*] = 0. \tag{3}$$

Dacă şi suprafeţele $u = \text{const}$ de pe varietăţile V şi V^* au planele tangente T_1 respectiv T_1^* semiparalele, atunci obţinem în punctele M şi M^*, unde aceste suprafeţe taie generatoarele g şi g^*, puncte în care H şi H^* sînt parţial paralele de dimensiunea considerată.

În adevăr, dacă T_1 şi T_1^* sînt semiparalele, atunci ele au un punct impropriu I_1 comun.

Cele două puncte improprii I_0 şi I_1 aparţin planelor improprii Π şi Π^*, conţinute respectiv în H şi H^*. Există o dreaptă improprie δ determinată de punctele I_0 şi I_1, care este comună planelor Π şi Π^*. Planul impropriu Π este determinat de dreapta δ şi punctul impropriu de pe g, iar planul Π^* de dreapta δ şi punctul impropriu de pe g^*. Astfel hiperplanele tangente H şi H^* au o dreaptă improprie comună δ şi sînt parţial paralele de dimensiune $\frac{2}{3}$.

Condiţia de semiparalelism a planelor T_1 şi T_1^* este dată de:

$$[\bar{a} + w\bar{\alpha}_v, \bar{\rho}_v^* + w^*\bar{\alpha}_v^*, \bar{\alpha}, \bar{\alpha}^*] = 0. \tag{4}$$

Dacă relaţia (3) este îndeplinită, atunci (4) determină o omografie între punctele generatoarelor g şi g^*, dată de ecuaţia:

$$\begin{aligned} ww^*[\bar{\alpha}_v, \bar{\alpha}_v^*, \bar{\alpha}, \bar{\alpha}^*] + w[\bar{\alpha}_v, \bar{\rho}_v^*, \bar{\alpha}, \bar{\alpha}^*] + \\ + w^*[\bar{a}, \bar{\alpha}_v^*, \bar{\alpha}, \bar{\alpha}^*] + [\bar{a}, \bar{\rho}_v^*, \bar{\alpha}, \bar{\alpha}^*] = 0. \end{aligned} \tag{5}$$

Punctele corespunzătoare în această omografie se corespund prin hiperplane tangente parţial paralele.

2. Ne propunem în continuare să determinăm elementele varietăţii V^*, cînd este dată varietatea V. Considerînd suprafaţa directoare S^* formată din puncte în care g^* taie hiperplanul H determinat de punctul ρ şi direcţiile $\bar{\rho}'$, $\bar{a}$, $\bar{\alpha}$, avem:

$$\bar{\rho}^* = \bar{\rho} + A(u, v)\bar{\rho}' + B(u, v)\bar{a} + C(u, v)\bar{\alpha},$$

în care $A(u, v)$, $B(u, v)$, $C(u, v)$ sînt funcţii arbitrare.

În problema propusă matricele funcţionale

$$L = \|\bar{\rho}', \bar{a}, \bar{\alpha}_u, \bar{\alpha}_v, \bar{\alpha}\| \quad \text{şi} \quad L^* = \|\bar{\rho}_u^*, \bar{\rho}_v^*, \bar{\alpha}_u^*, \bar{\alpha}_v^*, \bar{\alpha}^*\|$$

au rangul 3; reperul local ales este format din vectorii:

$$\bar{\rho}'(1, 0, 0, 0); \quad \bar{a}(0, 1, 0, 0); \quad \bar{\alpha}(0, 0, 1, 0); \quad \bar{\alpha}^*(0, 0, 0, 1).$$

Presupunînd introduse funcţii regulate în toate condiţiile impuse mai sus, obţinem următorul sistem de ecuaţii cu derivate parţiale pentru determinarea funcţiilor A, B, C:

$$\begin{aligned} A_v + s(u, v)C = m(u, v), \quad A_u + d(u)C = 0, \\ B_v + t(u, v)C = n(u, v), \quad C_v + p(v)C = 0. \end{aligned} \tag{6}$$

În aceste sisteme, d, m, n, p, t, s sînt funcţii determinate, depinzînd de elementele varietăţii V. Condiţia de integrabilitate pentru funcţia $A(u, v)$ este identic verificată $A_{uv} \equiv A_{vu}$.

Funcţia $A(u, v)$ satisface o ecuaţie Pfaff, deci soluţia ei depinde de o constantă arbitrară.

Funcţia C este de forma

$$C = f(u) \int e^{-p(v)} dv,$$

unde $f(u)$ este o funcţie arbitrară.

Funcţia B se determină din sistemul (6), ea depinzînd de o funcţie arbitrară $g(u)$.

Prin urmare, elementele varietăţii V^* sînt determinate în funcţie de elementele varietăţii V prin sistemul (6), ele depinzînd de două funcţii arbitrare şi de o constantă arbitrară; în punctele ce se corespund prin omografia (5) hiperplanele tangente sînt parţial paralele.

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UNE CORRESPONDENCE ENTRE VARIÉTÉS RÉGLÉES DANS S_4 ET LE SYSTÈME D'ÉQUATIONS DIFFÉRENTIELLES

(Résumé)

On établit une correspondence par hyperplans tangents partiellement parallèles entre deux variétés V et V^*, dont la dimension du parallélisme est 2/3. On trouve que les éléments de ces deux variétés satisfont le système (6) à l'aide de deux fonctions arbitraires et une constante.

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A FUNCTIONAL EQUATION ASSOCIATED WITH A NON-LINEAR RECURSION (II)

BY

S. G. GHURYE and WILLIAM S. GUSTIN

1. Introduction

In [1] we continued the investigation of a problem initiated in [2], namely the nature of sequences satisfying the recursion relation

$$x_{n-1}\, x_{n+1} = x_n - \alpha, \tag{1.1}$$

where α is an arbitrary real number. For each $\alpha \in (0, 1/4]$ there exists a sequence $\{x_n, n = 0, 1, 2, \ldots\}$ of positive numbers satisfying (1.1) with $x_0 = \alpha$ and there is only one such sequence for each α.

Relation (1.1) is a special case of the relation $x_{n+1} = g(x_{n-1}, x_n)$, where g is some function on R_2. Moving along the sequence from x_{n-1}, x_n to x_{n+1} is equivalent to moving from a point (x, y) to the point (y, z) through the transformation $z = g(x, y)$.

We are now going to investigate in detail the class of transformations involved in (1.1), using the geometrical aspect of the Theorem established in [1] .

2. Detailed Study of the Original Problem

The Theorem from [1] is of help in understanding the behaviour of solutions of recursions of the type we are concerned with. If $\{x_k, k = -m, -m+1, \ldots, n-1, n\}$ are numbers (all different from 0) satisfying $x_{k-1}x_{k+1} = g(x_k)$, then [1, (1.1)] implies that all the points (x_{k-1}, x_k) lie on the curve $F_0(x, y) = 0$ [1, (1.5)], with an appropriate choice of θ (the same for all k). If g is of the form stated in the Theorem, then the curves are quartics of the second degree in each variable; also, they are symmetric with respect to the equiangular line $y = x$. With the exception of the curve for which θ is given by $(\theta b_0 - c_1 b_1)(b_0^2 - c_0 b_2) = (b_0 b_1 - c_1 b_2)(c_1 b_0 - c_0 b_1)$, every line parallel to either coordinate axis meets each curve at two points or none. These properties enable us to use the curves to treat the recursion problem geometrically.

We shall now restrict ourselves to our original problem in which $g(x) = x - \alpha = \{-\alpha + (1-\alpha)x + x^2\}/(x+1)$, so that the family of orbits is given by the equations

$$(2.1) \qquad xy(x+y) + x^2 + y^2 + \theta xy + (1-\alpha)(x+y) - \alpha = 0, \ \theta \in R_1 .$$

A detailed study of these curves is easily made by the use of elementary methods. Every curve of the family passes through the points $(0, \alpha)$, $(\alpha, 0)$, $(0, -1)$, $(-1, 0)$. {This corresponds to the fact that if we take $x_1 = 0$ and $x_0 = \alpha$, all values of x_1 satisfy the recursion} and none passes through $(0, 0)$. Through every other point of the plane, there passes one and only one of these curves. The exceptional curve mentioned in [1] has $\theta = 2 - \alpha$ so that its equation is

$$(2.2) \qquad (x + y - \alpha)(x+1)(y+1) = 0.$$

The only other curve with a rectilinear branch is the one corresponding to $\theta = 3$ so that its equation is

$$(2.3) \qquad (x + y + 1)\{(x+1)(y+1) - 1 - \alpha\} = 0.$$

Given any point (x, y) which is not in the set $\{(0, 0), (0, \alpha), (\alpha, 0), (0, -1), (-1, 0)\}$, there is a unique θ such that $F(x, y) = 0$. There is one and only one other point on this curve with the same y-coordinate {unless $y = -1$} ; let (z, y) be this point. On substituting these values of θ, x, y, z in [1, (1.1)], the l. h. s. reduces to zero, so that, unless $x = z$ or $b(y) = 0$, we have $xz = g(y) = y - \alpha$. Hence the following geometric construction for the solution of (1.1): through the point (x_{n-1}, x_n), draw a straight line parallel to the x-axis ; through the intersection of this line with the equiangular line $y = x$, draw a straight line parallelt o the y-axis ; this line meets the orbit of (x_{n-1}, x_n) in the two points (x_n, x_{n-1}) and (x_n, x_{n+1}). Thus x_{n+1} is the y-coordinate of the point which is not the mirror image of (x_{n-1}, x_n) in the equiangular line. This construction is illustrated in Fig. 1.

The study of these curves is facilitated by a change of coordinates to take advantage of symmetry. Setting $u = x + y$ and $v = x - y$, (2.1) changes to

$$(2.4) \qquad \begin{aligned} (u + \theta - 2)v^2 &= u^3 + (\theta + 2)u^2 + 4(1-\alpha)u - 4\alpha = \\ &= u^2(u + \theta - 2) + 4(u - \alpha)(u+1). \end{aligned}$$

The earlier statement that (2.2) and (2.3) are the only two curves of the family which have recti-linear branches is now readily verified. We also see that every line parallel to $x + y = 0$ meets any curve other than (2.2) and (2.3) in either two points or none. Now (2.2) has three double points $\{(-1, -1), (-1, 1+\alpha), (1+\alpha, -1)\}$, (2.3) has none, and other curves of the family can have double points only on the equiangular line, and a curve has at most one double point. It is easy to see that only two other curves of the family have double points, namely those corresponding to $\theta = \theta' = -\{1 + \xi' + 4(\xi')^2\}/\xi'$ and $\theta = \hat{\theta} = -\{1 + \xi + 4\xi^2\}/\xi$, their double points being (ξ', ξ') and (ξ, ξ) respectively, where $\xi < \xi'$ are the zeroes of

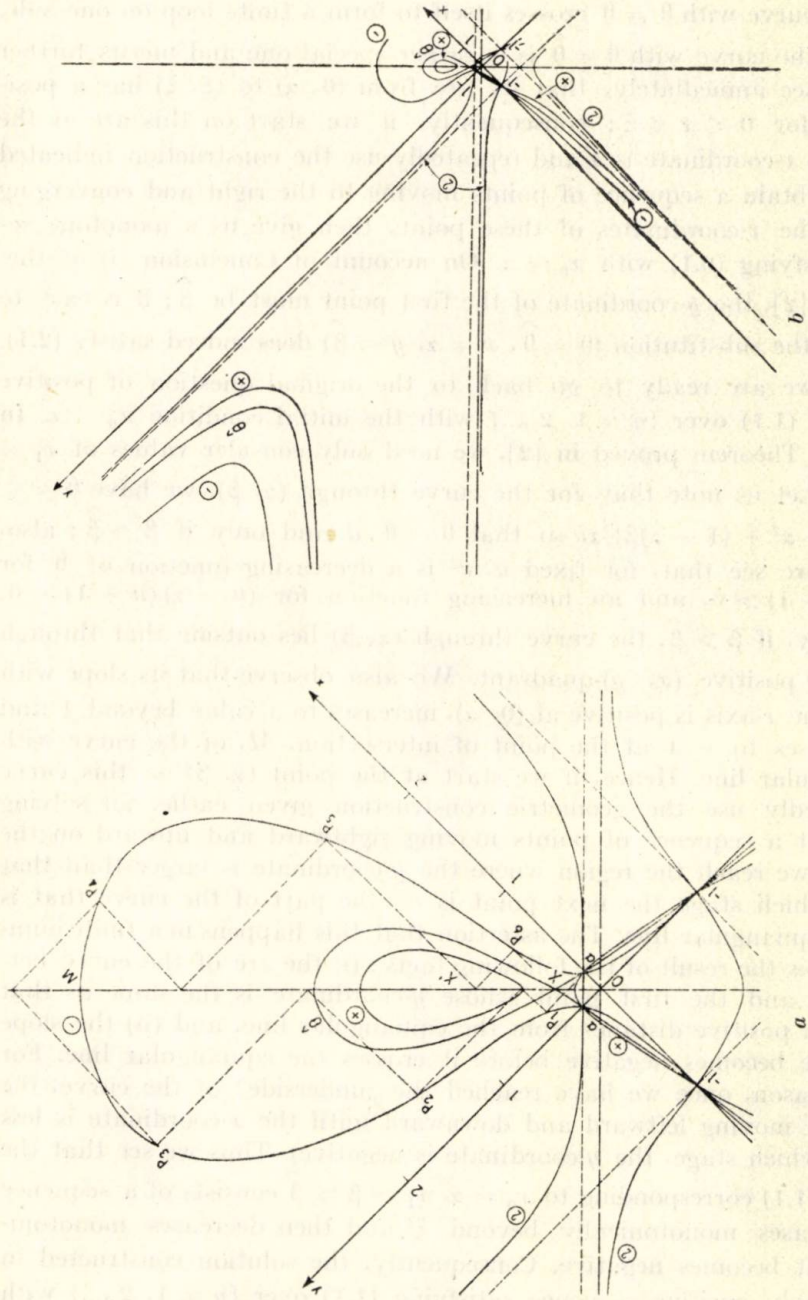

Fig. 1. — Orbits associated with the recursion (1.1) with $\alpha=3/16$; a — large-scale picture of the central region which is crucial for the discussion of positive solutions ; b — intended to give a general idea over the whole plane.

X and X' are the points (ξ, ξ) and (ξ', ξ'), where $\xi = 1/4$ and $\xi'=3/4$; the critical value $\hat{\theta}$ of θ is -6 and $\theta'=-16/3$. The values of θ for the curves indicated by symbols are as follows : $\hat{\theta}:\theta=\hat{\theta}=-6$; „—" : $\theta=-8$, typical of $\theta<\hat{\theta}$; „+" : $\theta=-5.7$, typical of $\theta\in[\hat{\theta},\theta']$; „2" : $\theta=2$, as an example of shapes different from those observed for $\theta\leqslant\theta'$.

$x^2 - x + \alpha = 0$. The first is an isolated point of its curve, and at the second the curve with $\theta = \theta$ crosses itself to form a finite loop on one side.

Thus the curve with $\theta = \hat{\theta}$ is a rather special one and merits further study. We see immediately, that its arc from $(0, \alpha)$ to (ξ, ξ) has a positive slope for $0 \leqslant x \leqslant \xi$; consequently, if we start on this arc at the point whose x-coordinate is α and repeatedly use the construction indicated above, we obtain a sequence of points moving to the right and converging to (ξ, ξ). The x-coordinates of these points then give us a monotone sequence satisfying (0.1) with $x_0 = \alpha$. On account of Conclusion (i) of the Theorem in [2], the y-coordinate of the first point must be $\hat{\beta}$; it is easy to verify that the substitution $(\theta = \hat{\theta}, x = \alpha, y = \hat{\beta})$ does indeed satisfy (2.1).

Now we are ready to go back to the original question of positive solutions of (1.1) over $\{n = 1, 2, ...\}$ with the initial condition $x_0 = \alpha$. In view of the Theorem proved in [2], we need only consider values of $x_1 = \beta > \hat{\beta}$. Let us note that, for the curve through (α, β), we have $\theta = -\{1 - \alpha + \alpha^2 + (1 + \alpha)\beta\}/\alpha$, so that $\theta < \hat{\theta}$ if and only if $\beta > \hat{\beta}$; also, from (2.4) we see that, for fixed u, v^2 is a decreasing function of θ for $(u - \alpha)(u + 1) > 0$, and an increasing function for $(u - \alpha)(u + 1) < 0$. Consequently, if $\beta > \hat{\beta}$, the curve through (α, β) lies outside that through $(\alpha, \hat{\beta})$ in the positive (x, y)-quadrant. We also observe that its slope with respect to the x-axis is positive at $(0, \alpha)$, increases to a value beyond 1 and then decreases to -1 at the point of intersection, M, of the curve with the equiangular line. Hence, if we start at the point (α, β) on this curve and repeatedly use the geometric construction given earlier for solving (1.1), we get a sequence of points moving rightward and upward on the curve until we reach the region where the y-coordinate is larger than that of M, at which stage the next point is on the part of the curve that is under the equiangular line. The assertion that this happens in a finite number of steps is the result of the following facts : (i) the arc of the curve between $(0, \alpha)$ and the first point whose y-coordinate is the same as that of M is at a positive distance from the equiangular line, and (ii) the slope of the curve becomes negative before it crosses the equiangular line. For the same reason, once we have reached the „underside" of the curve, the points begin moving leftward and downward until the x-coordinate is less than α (at which stage, the y-coordinate is negative). Thus we see that the solution of (1.1) corresponding to $x_0 = \alpha$, $x_1 = \beta > \hat{\beta}$ consists of a sequence which increases monotonically beyond ξ' and then decreases monotonically until it becomes negative. Consequently, the solution constructed in [2] is the only positive sequence satisfying (1.1) over $\{n = 1, 2, ...\}$ with $x_0 = \alpha$.

Finally, should the question be raised about positive solutions of (1.1) without the initial condition $x_0 = \alpha$, the answer is provided immediately : They are precisely those solutions which reside on the orbitloops contained within the loop of the critical curve.

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O ECUAŢIE FUNCŢIONALĂ ASOCIATĂ CU O RELAŢIE DE RECURENŢĂ NELINIARĂ

(Rezumat)

Utilizînd aspecte geometrice ale unei teoreme stabilite în [1] se aprofundează studiu şirurilor care satisfac relaţia de recurenţă (1.1).

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ÜBER EINIGE ÜBERBESTIMMTE SYSTEME PARTIELLER DIFFERENTIALGLEICHUNGEN

VON

NIKOLAOS APOSTOLATOS

0. Schon in der Arbeiten [1], [2], [3] wurden überbestimmte Systeme partieller Differentialgleichungen untersucht. In [2] wurde gezeigt, daß solche Systeme unter gewissen Voraussetzungen auf die sog. kanonische Form übergeführt werden können. Für Systeme in kanonischer Form wurden in [2] außerdem Integrabilitätsbedingungen angegeben.

In der vorliegenden Arbeit werden zwei Klassen überbestimmter Systeme partieller Differentialgleichungen angegeben, deren Vollständigkeit nicht durch Überführung auf die kanonische Form, sondern mit Hilfe einer direkten Methode gezeigt wird. Neben der Vollständigkeit wird für diese beiden Klassen von überbestimmten Systemen mit den Anfangsbedingungen (1.3) bzw. (2.3) auch die Eindeutigkeit der Lösung gezeigt.

1. Es sei nun das folgende Anfangswertproblem gegeben:

$$\frac{\partial^{n_1} f}{\partial x^{n_1}} + P_1(x)\frac{\partial^{n_1-1} f}{\partial^{n_1-1} x} + \dots + P_{n_1-1}(x)\frac{\partial f}{\partial x} + P_{n_1}(x) f = 0, \tag{1.1}$$

$$\frac{\partial^{n_2} f}{\partial y^{n_2}} Q_1(y)\frac{\partial^{n_2-1} f}{\partial y^{n_2-1}} + \dots + Q_{n_2-1}(y)\frac{\partial f}{\partial y} + Q_{n_2}(y) f = 0, \tag{1.2}$$

$$\left[\frac{\partial q_f}{\partial x^{q_1}\,\partial y^{q_2}}\right]_{\substack{x=x^0\\ y=y^0}} = f_{q_1 q_2}, \qquad \begin{bmatrix} q = q_1 + q_2 \\ q_1 = 0, 1, \dots, n_1 - 1 \\ q_2 = 0, 1, \dots, n_2 - 1 \end{bmatrix}. \tag{1.3}$$

Dabei sind die Koeffizienten $P_i(x)$ $(i = 1, 2, \dots, n_1)$ bzw. $Q_j(y)$ $(j = 1, 2, \dots, n_2)$ in einer Umgebung von x_0 bzw. y_0 analytische Funktionen von x bzw. y. Ohne weiteres könnte man dieses System in die kanonische Form transformieren und anschließend die Erfüllung der entsprechenden Integrabilitätsbedingungen untersuchen. Dies wäre allerdings eine sehr mühsame Angelegenheit. Es soll also hier mit Hilfe einer direkten Methode, welche nicht nur viel kürzer sondern auch viel eleganter ist gezeigt werden, daß dieses überbestimmte System vollständig integrabel ist (vgl. dazu [2]) und zu jedem Anfangswertsystem (1.3) genau eine Lösung besitzt.

Sei $\varphi_\nu(y)$ $(\nu = 1, 2, \ldots, n_2)$ ein beliebiges, aber festes fundamentales Lösungssystem der gewöhnlichen linearen Differentialgleichung, welche aus (1.2) entsteht, wenn man die dort auftretenden partiellen Ableitungen durch totale ersetzt, so lautet die allgemeine Lösung von (1.2)

$$f(x, y) = \sum_{\nu=1}^{n_2} \varphi_\nu(x)\, \psi_\nu(y). \tag{1.4}$$

Die $\varphi_\nu(x)$ $(\nu = 1, 2, \ldots, n_2)$ sind zunächst ganz beliebige Funktionen von x. Verlangt man aber von der Funktion (1.4) auch die Erfüllung der Differentialgleichung (1.1), so müssen die $\varphi_\nu(x)$ $(\nu = 1, 2, \ldots, n_2)$ n_1-mal stetig differenzierbare Funktionen von x sein. Wird dies vorausgesetzt dann erhält man durch Einsetzen der Funktion (1.4) in (1.1):

$$\sum_{\nu=1}^{n_2} \psi_\nu(y) \left[\frac{d^{n_1}\varphi_\nu(x)}{dx^{n_1}} + P_1(x) \frac{d^{n_1-1}\varphi_\nu(x)}{dx^{n_1-1}} + \ldots + P_{n_1}(x)\, \varphi_\nu(x) \right] = 0.$$

Da die Funktionen $\psi_\nu(y)$ $(\nu = 1, 2, \ldots, n_2)$ linear unabhängig sind, muß gelten:

$$\frac{d^{n_1}\varphi_\nu(x)}{dx^{n_1}} + P_1(x) \frac{d^{n_1-1}\varphi_\nu(x)}{dx^{n_1-1}} + \ldots + P_{n_1}(x)\, \varphi_\nu(x), 0, \quad \nu = 1, \ldots, n_2. \tag{1.5}$$

Daraus läßt sich folgern, daß jede der Funktionen $\varphi_\nu(x)$ Lösung der gewöhnlichen Differentialgleichung sein muß, welche aus (1.1) entsteht, wenn man die partiellen Ableitungen durch totale ersetzt. Setzt man nun die Funktion (1.4) in (1.3) ein, so ergibt sich:

$$\sum_{\nu=1}^{n_2} \varphi_\nu^{(q_1)}(x_0) \psi_\nu^{(q_2)}(y_0) = \mathring{f}_{q_1 q_2}, \qquad \left[\begin{array}{l} q_1 = 0, 1, \ldots, n_1 - 1 \\ q_2 = 0, 1, \ldots, n_2 - 1 \end{array} \right]. \tag{1.6}$$

Bezeichnet man mit $w(\psi_\nu(y))$ die Wronskische Determinante der Funktionen $\psi_\nu(y)$ $(\nu = 1, 2, \ldots, n_2)$, so ist sie, wegen der linearen Unabhängigkeit der Funktionen $\psi_\nu(y)$, von Null verschieden. Darüberhinaus läßt sich für jedes $q_1 = 0, 1, \ldots, n_1 - 1$ das entsprechende System aus (1.6) eindeutig nach den Größen $\varphi_\nu^{(q_1)}(x_0)$ $(\nu = 1, 2, \ldots, n_2;\ q_1 = 0, 1, \ldots, n_1 - 1)$ auflösen.

Mit Hilfe der gefundenen Anfangswerte $\varphi_\nu^{(q_1)}(x_0)$ lassen sich die Funktionen $\varphi_\nu(x)$ mittels der Differentialgleichung (1.5) eindeutig definieren.

Damit ist also bewiesen, daß das überbestimmte System (1.1) und (1.2) für jedes beliebige Anfangswertsystem (1.3) genau eine Lösung besitzt. Diese Lösung ist eine analytische Funktion von x und y. Damit ist die Behauptung völlig bewiesen.

2. Als eine andere Klasse von überbestimmten Systemen partieller Differentialgleichungen soll jetzt das folgende Anfangswertproblem betrachtet werden.

$$\sum_{k=0}^{n} \sum_{l=0}^{k} A_{l,\,k-l}(y) \frac{\partial^k f}{\partial x^l\, \partial y^{k-l}} = 0, \quad (\text{mit } A_{on}(y) \equiv 1), \tag{2.1}$$

$$\frac{\partial^m f}{\partial x^m} + a_1 \frac{\partial^{m-1} f}{\partial x^{m-1}} + a_2 \frac{\partial^{m-2} f}{\partial x^{m-2}} + \dots + a_m f = 0, \tag{2.2}$$

$$\left[\frac{\partial q_f}{\partial x^{q_1} \partial y^{q_2}} \right]_{\substack{x=x^0 \\ y=y^0}} = \overset{\circ}{f}_{q_1 q_2}, \quad \left[\begin{array}{l} q = q_1 + q_2 \\ q_1 = 0, 1, \dots, m-1 \\ q_2 = 0, 1, \dots, n-1 \end{array} \right]. \tag{2.3}$$

Es seien $A_{l,k-l}(y)$ in einer Umbgebung von y_0 analytische Funktionen von y und a_i beliebige konstante komplexe Zahlen. Es soll auch hier mit der unter 1. angeführten Methode gezeigt werden, daß durch (2.1), (2.2) und (2.3) genau eine Lösung festgelegt wird. Die allgemeine Lösung von (2.2) lautet :

$$\begin{aligned} f(x, y) &= \sum_{\alpha=1}^{\nu} (\varphi_{0\mu_\alpha}(y) + x\varphi_{1\mu_\alpha}(y) + \dots + x^{\mu_\alpha} \varphi_{\mu_\alpha\mu_\alpha}(y)) e^{r_\alpha x} = \\ &= \sum_{\alpha=1}^{\nu_\alpha} \sum_{\lambda=0}^{\mu_\alpha} x^\lambda e^{r_\alpha x} \varphi_{\lambda\mu_\alpha}(y). \end{aligned} \tag{2.4}$$

Dabei sind $r_1, r_2, \dots, r_\nu$ die Nullstellen des charakteristischen Polynoms $u^m + a_1 u^{m-1} + a_2 u^{m-2} \dots + a_m$ und $\mu_1+1, \mu_2+1, \dots, \mu_\nu+1$ deren Vielfachheit. Es gilt also :

$$\mu_1 + \mu_2 + \dots + \mu_\nu = m - \nu, \tag{2.5}$$

$\varphi_{\lambda\mu_\alpha}(y)$ ($\lambda = 0, 1, \dots, \mu_\alpha$; $\alpha = 1, 2, \dots, \nu$) sind zunächst ganz beliebige Funktionen von y. Wenn die Funktion (2.4) auch Lösung von (2.1) sein soll, müssen die $\varphi_{\lambda\mu_\alpha}(y)$ zusätzlich n-mal stetig differenzierbare Funktionen von y sein. Unter dieser Voraussetzung liefert die Gleichung (2.1), wenn man (2.4) einsetzt :

$$\sum_{k=0}^{n} \sum_{l=0}^{k} A_{l,k-l}(y) \sum_{\alpha=1}^{\nu} \sum_{\lambda=0}^{\mu_\alpha} \varphi_{\lambda\mu_\alpha}^{(k-l)}(y) (x^\lambda e^{r_\alpha x})^{(l)} = 0$$

oder

$$\sum_{k=1}^{n} \sum_{l=0}^{k} A_{l,k-l}(y) \sum_{\alpha=1}^{\nu} \sum_{\lambda=0}^{\mu_\alpha} \varphi_{\lambda\mu_\alpha}^{(k-l)}(y) \sum_{q=0}^{\min(l,\lambda)} \binom{l}{q} (x^\lambda)^{(q)} r_\alpha^{l-q} e^{r_\alpha x} = 0. \tag{2.6}$$

Aus (2.3) und (2.4) erhält man aber :

$$\sum_{\alpha=1}^{\nu} [\varphi_{0\mu_\alpha}^{(q_2)}(y) (e^{r_\alpha x})^{(q_1)} + \dots + \varphi_{\mu_\alpha\mu_\alpha}^{(q_2)}(y) (x^{\mu_\alpha} e^{r_\alpha x})^{(q_1)}]_{\substack{x=x^0 \\ y=y^0}} = \overset{\circ}{f}_{q_1 q_2}, \tag{2.7}$$

mit $q_1 = 0, 1, \dots, m-1$; $q_2 = 0, 1, \dots, n-1$.

Bezeichnet man mit $w(x)$ die Wronskische Determinante der Funktionen $e^{r_1 x}, xe^{r_1 x}, \dots, x^{\mu_1} e^{r_1 x}, e^{r_2 x}, xe^{r_2 x}, \dots, x^{\mu_2} e^{r_2 x}, \dots, e^{r_\nu x}, xe^{r_\nu x}, \dots, x^{\mu_\nu} e^{r_\nu x}$ so ist sie, wegen der linearen Unabhängigkeit dieser Funktionen von Null verschieden.

Betrachtet man nun die Gleichungen (2.7) für jedes feste $q_2 \in \{0,1,\ldots,$ $,\ldots, n-1\}$ als ein lineares Gleichungssystem mit den unbekannten Größen $\varphi^{(q_2)}_{\lambda\mu_\alpha}(y_0)$, $\lambda = 0, 1, \ldots, \mu_\alpha$, $\alpha = 1, 2, \ldots, \nu$ and löst man es nach diesen Größen auf, so bekommt man

$$\varphi^{(q_2)}_{\lambda\mu_\alpha}(y_0) = \frac{w^{q_2}_{\lambda\mu_\alpha}}{w(x_0)}, \tag{2.8}$$

mit $q_2 = 0, 1, \ldots, n-1$, $\lambda = 0, 1, \ldots, \mu_\alpha$; $\alpha = 1, 2, \ldots, \nu$.

Dabei ist $w^{q_2}_{\lambda\mu_\alpha}$ die Determinante, welche aus $w(x_0)$ entsteht, wenn man die Spalte

$$\begin{bmatrix} x^\lambda e^{r_\alpha x} \\ (x^\lambda e^{r_\alpha x})' \\ \vdots \\ (x^\lambda e^{r_\alpha x})^{(m-1)} \end{bmatrix} \quad \text{durch die Spalte} \quad \begin{vmatrix} \mathring{f}_{0q_2} \\ \mathring{f}_{1q_2} \\ \vdots \\ \mathring{f}_{1q_2} \end{vmatrix} \quad \text{ersetzt.}$$

Da die Funktionen $e^{r_\alpha x}$ $(\alpha = 1, 2, \ldots, \nu)$ linear unabhängig sind, erhält man als notwendige und hinreichende Bedingungen für das Bestehen der Gleichungen (2.6) folgende Beziehungen:

$$\sum_{k=0}^{n} \sum_{l=0}^{k} A_{l,\,k-l}(y) \sum_{\lambda=0}^{\mu_\alpha} \varphi^{(k-l)}_{\lambda\mu_\alpha}(y) \sum_{q=0}^{\min(l,\lambda)} \binom{l}{q} (x^\lambda)^{(q)} r_\alpha^{l-q} = 0$$

für $\alpha = 1, 2, \ldots, \nu$.

Und wegen der linearen Unabhängigkeit der Funktionen $1, x, x^2, \ldots,$ ergibt sich weiter

$$\sum_{k=0}^{n} \sum_{l=0}^{k} A_{l,\,k-l}(y) \sum_{q=0}^{\min(l,\mu_\alpha-p)} \binom{l}{q} \frac{(p+q)!}{p!} r_\alpha^{l-q} \varphi^{(k-l)}_{q+p,\mu_\alpha}(y) = 0 \tag{2.9}$$

für $p = 0, 1, \ldots, \mu_\alpha$; $\alpha = 1, 2, \ldots, \nu$.

Die $(\mu_1 + 1) + (\mu_2 + 1) + \ldots + (\mu_\nu + 1) =$ (nach (2.5)) $= m$ unbekannten Funktionen $\varphi_{\lambda\mu_\alpha}(y)$ erfüllen also ein gewöhnliches Differentialgleichungssystem mit genau so vielen Gleichungen. Nun soll gezeigt werden, daß dieses System genau eine, und zwar eine analytische Lösung besitzt, die Anfangsbedingungen (2.8) erfüllt. Dies bedeutet aber, daß das ursprüngliche Anfangswertproblem (2.1), (2.2) und (2.3) genau eine und zwar eine analytische Lösung besitzt. Zum Beweis der Behauptung geht man folgendermaßen vor. Für jedes bestimmte $\{\alpha \in 1, 2, \ldots, \nu\}$ erhält man aus (2.9) ein gewöhnliches Differentialgleichungssystem mit $\mu_\alpha + 1$ unbekannten Funktionen $\varphi_{0\mu_\alpha}(y), \varphi_{1\mu_\alpha}(y), \ldots, \varphi_{\mu_\alpha\mu_\alpha}(y)$ und genau so vielen Differentialgleichungen n-ter Ordnung.

Löst man jede dieser Differentialgleichungen nach der höchsten auftretenden Ableitung auf, so bekommt man (da $A_{on}(y) \equiv 1$ ist)

$$\varphi^{(n)}_{p\mu_\alpha}(y) = -\sum_{k=0}^{n'}\sum_{l=0}^{k'} A_{l,\,k-l}(y) \sum_{q=0}^{\min(l,\mu_\alpha-p)} \binom{l}{q} \frac{(p+q)!}{p!} r_\alpha^{l-q}\, \varphi^{(k-l)}_{q+p,\mu_\alpha}(y) = 0$$

für $p = 0, 1, \ldots, \mu_\alpha$. Dabei bedeutet der Strich (′) bei den Summanden, daß das Glied mit $k=n$, $l = 0$, $q = 0$ ausgeschlossen werden muß.

Wendet man die Transformationen

$$(2.10)\quad \begin{cases} \varphi_{p\mu_\alpha}(y) = \varphi_{0p\mu_\alpha}(y), \\ \varphi'_{p\mu_\alpha}(y) = \varphi'_{0p\mu_\alpha}(y) = \varphi_{1p\mu_\alpha}(y), \\ \varphi''_{p\mu_\alpha}(y) = \varphi'_{1p\mu_\alpha}(y) = \varphi_{2p\mu_\alpha}(y), \\ \varphi'''_{p\mu_\alpha}(y) = \varphi'_{2p\mu_\alpha}(y) = \varphi_{3p\mu_\alpha}(y), \\ \ldots\ldots \text{ u.s.w. } \ldots\ldots \end{cases} \qquad \begin{bmatrix} p = 0, 1, \ldots, \mu_\alpha, \\ \alpha = 1, 2, \ldots, \nu \end{bmatrix},$$

an, so nimmt dieses Differentialgleichungssystem folgende Gestalt an:

$$(2.11)\quad \begin{aligned} &\varphi'_{0p\mu_\alpha}(y) = \varphi_{1p\mu_\alpha}(y), \\ &\varphi'_{1p\mu_\alpha}(y) = \varphi_{2p\mu_\alpha}(y), \\ &\ldots\ldots\ldots\ldots \\ &\varphi'_{n-2,p\mu_\alpha}(y) = \varphi_{n-1,p\mu_\alpha}(y), \\ &\varphi'_{n-1,p\mu_\alpha}(y) = -\sum_{k=0}^{n'}\sum_{l=0}^{k'} A_{l,\,k-l}(y) \sum_{q=0}^{\min(l,\mu_\alpha-p)} \binom{l}{q} \frac{(p+q)!}{p!} r_\alpha^{l-q}\, \varphi_{k-l,\,q+p,\,\mu_\alpha}(y) \end{aligned}$$

für $p = 0, 1, \ldots, \mu_\alpha$; $\alpha = 1, 2, \ldots, \nu$.

Dieses ist für jedes feste α und p ein gewöhnliches lineares Gleichungssystem 1. Ordnung, dessen unbekannte Funktionen $\varphi_{\eta p\mu_\alpha}(y)$ $(\eta = 0, 1, \ldots, n-1)$ wegen (2.10) und (2.8) folgende Anfangsbedingungen erfüllen:

$$(2.12)\qquad \varphi_{\eta p\mu_\alpha}(y_0) = \varphi^{(\eta)}_{p\mu_\alpha}(y_0), \quad (\eta = 0, 1, \ldots, n-1).$$

Da die Koeffizienten von (2.11) analytische Funktionen sind, folgt unmittelbar, daß das System (2.11) genau eine und zwar eine analytische Lösung besitzt, die die Anfangsbedingungen (2.12) erfüllt. Damit ist die Behauptung völlig bewiesen.

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ASUPRA UNOR SISTEME DE ECUAŢII CU DERIVATE PARŢIALE SUPRADETERMINATE

(Rezumat)

Se studiază printr-o metodă directă compatibilitatea a două clase de sisteme de ecuaţii diferenţiale cu derivate parţiale supradeterminate, metodă care evită reducerea lor la forma canonică.

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ASUPRA UNOR METODE NUMERICE PENTRU PROBLEMA GOURSAT

DE

N. GRĂDINARU şi GEORGETA TEODORU

În această lucrare se arată cum se pot determina soluţii aproximative ale problemei Goursat pentru ecuaţii hiperbolice de ordinul al doilea, folosind metode de aproximare a soluţiei problemei Cauchy corespunzătoare în sensul dat de [1].

Fie un spaţiu funcţional local convex secvenţial complet X şi un operator liniar închis L definit în X, cu domeniul $D(L)$ şi codomeniul $R(L)$, cu $\overline{D(L)} = X$.

Fie $\Delta = [0, R] \times [0, S]$ cu R, S numere pozitive oarecari. Problema Goursat considerată aici, ca şi în [1], constă în determinarea unei funcţii $u(r, s; \cdot)$ definită pentru $(r, s) \in \Delta$ cu valori în X, cu $u(r, s; \cdot) \in C^2(\Delta, X)$ şi $u(r, s; \cdot) \in D(L)$ pentru $(r, s) \in \Delta$, care să satisfacă ecuaţia

$$\frac{\partial^2 u}{\partial r \partial s} = Lu \tag{1}$$

şi condiţia

$$u(0, s; \cdot) = u(r, 0; \cdot) = \varphi(\cdot), \tag{2}$$

unde $\varphi(\cdot)$ este un element arbitrar dat din X.

Acestei probleme i se asociază următoarea problemă Cauchy: determinarea unei funcţii $v(t; .) \in C^2([0, T], X)$ pentru care $v(t; .) \in D(L)$, satisfăcînd ecuaţia:

$$\frac{\partial^2 v}{\partial t^2} = Lv \tag{3}$$

şi condiţiile

$$v(0; \cdot) = \varphi(.), \qquad \frac{\partial v}{\partial t}(0; \cdot) = 0. \tag{4}$$

În [1] se demonstrează următoarea

T e o r e m ă. *Dacă v este soluţie a problemei Cauchy* (3), (4) *atunci u dată de*

$$u(r,s\,;\ .) = \frac{2}{\pi}\int_0^{\pi/2} v\,(2\sqrt{rs}\sin\theta\,;\ \cdot\,)\,d\theta \tag{5}$$

este soluţie a problemei Goursat (1), (2). *Invers, dacă* (1), (2) *are soluţia unică u, atunci v dată de*

$$v(t\,;\cdot) = \frac{d}{dt}\int_0^{\pi/2} t\sin\theta\, u\left(\frac{t\sin\theta}{2}, \frac{t\sin\theta}{2};\ \cdot\right)d\theta \tag{6}$$

este soluţie a problemei (3), (4).

Ţinînd cont de această echivalenţă putem deduce, în cazuri concrete, procedee de calcul a soluţiei aproximative a problemei (1), (2), folosind metode numerice pentru problema (3), (4). În cele ce urmează vom indica două astfel de metode.

I. Considerăm în Δ o reţea dreptunghiulară de noduri $\Omega_{pq} = \{(r_i, s_j)\,;\ i = 0, 1, 2, \ldots, p\,;\ j = 0, 1, 2, \ldots, q\}$, unde $r_i = ih$, $s_j = jl$, h şi l fiind paşii reţelei, respectiv în direcţiile r şi s, iar $p = R/h$, $q = S/l$.

Valorile exacte ale soluţiei problemei (1), (2) în nodurile (r_i, s_j) vor fi

$$u_{ij} = \frac{2}{\pi}\int_0^{\pi/2} v(2\sqrt{r_i s_j}\sin\theta\,;\ \cdot\,)\,d\theta. \tag{7}$$

De obicei schemele cu diferenţe pentru problema (3), (4) furnizează valorile funcţiei v pe o reţea de puncte din intervalul $[0, T]$, şi o reţea de puncte pentru variabila x a funcţiilor care constituie elementele spaţiului funcţional X.

Se impune astfel aplicarea unei formule de cuadratură pentru integrala ce apare în (7) pentru a obţine o soluţie aproximativă a problemei (1), (2) folosind o soluţie aproximativă a problemei (3), (4).

Fie o partiţie $0 = \theta_0 < \theta_1 < \cdots < \theta_k < \cdots < \theta_n = \pi/2$ a intervalului $[0, \pi/2]$. Din (7) obţinem:

$$u_{ij} = \frac{2}{\pi}\sum_{k=0}^{n} c_k\, v\,(2\sqrt{r_i s_j}\sin\theta_k\,;\ x_0) + \rho_n\,, \tag{8}$$

unde x_0 este un punct arbitrar dat pentru moment fixat, c_k sînt coeficienţii formulei de cuadratură, iar ρ_n restul acestei formule. Fie $t_{ijk} = 2\sqrt{ihjl}\sin\theta_k$ $(i = 0, 1, \ldots, p\,;\ j = 0, 1, \ldots, q\,;\ k = 0, 1, \ldots, n)$. Se observă că $t_{ijk} \in [0, T]$ unde $T = t_{pqn} = 2\sqrt{RS}$.

Luăm pasul Δt al reţelei pentru (3), (4) astfel încît

$$\Delta t \leqslant \min\{|t_{ijk} - t_{i'j'k'}|\,;\ i, i' = 0, 1, \ldots p\,;\ j, j' = 0, 1, \ldots q\,;\ k, k' = 0, 1, \ldots, n\,;\ (i, j, k) \neq (i', j', k')\}.$$

Aproximăm valoarea $v(t_{ijk}; x_0)$ prin valoarea aproximativă $v_{ijk}(x_0)$ a funcţiei v dată de schema cu diferenţe pentru (3), (4) în nodul reţelei corespunzătoare situat la distanţa cea mai mică de punctul t_{ijk}.

Notăm cu $\tilde{u}^n_{ij}$ valoarea aproximativă pentru $u(r_i, s_j; x_0)$ dată de expresia

$$\tilde{u}^n_{ij} = \frac{2}{\pi} \sum_{k=0}^{n} c_k \, v_{ijk}(x_0). \tag{9}$$

Se pune deci problema convergenţei şirului $\tilde{u}^n_{ij}$ la $u(r_i, s_j; x_0)$. Din relaţiile (8) şi (9) se obţine

$$u_{ij} - \tilde{u}^n_{ij} = \frac{2}{\pi} \sum_{k=0}^{n} c_k \, [v(t_{ijk}; x_0) - v_{ijk}(x_0)] + \rho_n. \tag{10}$$

De aici rezultă că ordinul de convergenţă a şirului $\tilde{u}^n_{ij}$ este acelaşi cu ordinul de convergenţă a şirului $v_{ijk}(x_0)$. De exemplu, în cazul problemei

$$\frac{\partial^2 v}{\partial t^2} = \frac{\partial^2 v}{\partial x^2}, \tag{11}$$

$$v(0; x) = \varphi(x), \quad \frac{\partial v}{\partial t}(0; x) = 0, \tag{12}$$

$$(t, x) \in G = \{(t, x) \in R^2; \; t \geqslant 0, \; -\infty < x < \infty\}$$

considerăm schema cu diferenţe

$$\frac{v_{\alpha+1,\beta} - 2v_{\alpha\beta} + v_{\alpha-1,\beta}}{(\Delta t)^2} - \frac{v_{\alpha,\beta+1} - 2v_{\alpha\beta} + v_{\alpha,\beta-1}}{(\Delta x)^2} = 0, \tag{11'}$$

cu $\Delta t = \gamma \Delta x$ $(\gamma \leqslant 1)$ şi $v_{\alpha\beta} \approx v(\alpha\Delta t, \beta\Delta x)$, $\alpha = 1, 2, \ldots$; $\beta = 0, \pm 1, \pm 2, \ldots$

$$v_{0\beta} = \varphi(\beta\Delta x) = \varphi_\beta, \quad v_{1\beta} = \frac{1}{2}[\varphi_{\beta+1} + \varphi_{\beta-1}]. \tag{12'}$$

În [2], în cazul cînd v admite derivate parţiale mărginite pînă la ordinul al patrulea inclusiv, se obţine evaluarea

$$|v_{\alpha\beta} - v(\alpha\Delta t, \beta\Delta x)| \leqslant \frac{(\Delta x)^2}{12}(M_4 \Delta x + 2M_3)\, t_0 + \frac{(\Delta x)^2}{12} t_0^2 M_4 \tag{13}$$

pentru punctul de coordonate (t_0, x_0), cu $t_0 = \alpha\Delta t$, $x_0 = \beta\Delta x$, unde $M_k = \max\limits_G \{|v^{(k)}_{t^k}|, \; |v^{(k)}_{x^k}|\}$.

II. Se constată uşor că teorema din [3] rămîne valabilă şi în cazul problemei

$$Lv \equiv \frac{\partial^2 v}{\partial t^2} - \frac{\partial}{\partial x_j}\left(a_{ij}(x)\frac{\partial v}{\partial x_i}\right) + b_i(x)\frac{\partial v}{\partial x_i} + b_0(x)\frac{\partial v}{\partial t} + c(x)\, v = f(x), \tag{14}$$

$$(15) \qquad v(0\,;\,x) = \Phi(x), \quad \frac{\partial v}{\partial t}\,(0\,;\,x) = 0,$$

$$(i, j = 1, 2, \ldots n), \quad x = (x_1, x_2, \ldots, x_n) \in \Omega \subset R^n, \qquad t \in [0, T],$$

unde Φ este o funcţie reală continuă, cu derivate parţiale continui pînă la ordinul IV inclusiv în domeniul Ω şi cu suportul compact inclus în Ω.

Presupunem că L şi f satisfac condiţiile teoremei din [3], unde se arată că prelungirile $v_{\Delta t, \Delta x}(t\,;\,x)$ ale soluţiei aproximative $v_{\Delta t, \Delta x}(\alpha\Delta t, \beta\Delta x)$, date de schema cu diferenţe din [3], ($\beta\Delta x = (\beta_1\Delta x, \beta_2\Delta x, \ldots, \beta_n\Delta x)$, converg slab în $L^2([0, T] \times \Omega)$ la soluţia exactă a problemei (14), (15) pentru Δt, $\Delta x \to 0$; ($\Delta t \leqslant \gamma\Delta x$, γ este o anumită constantă).

Rezultă atunci că funcţiile

$$(16) \qquad u_{\Delta t, \Delta x}(r, s\,;\,x) = \frac{2}{\pi}\int_0^{\pi/2} v_{\Delta t, \Delta x}(2\sqrt{rs}\sin\theta\,;\,x)\,d\theta$$

aproximează soluţia lui (1), (2) în sensul că şirul $T_\varphi u_{\Delta t, \Delta x}(r, s\,;\,x)$ tinde la $T_\varphi u(r, s\,;\,x)$ pentru Δt, $\Delta x \to 0$ ($\Delta t \leqslant \gamma\Delta x$), unde operatorul T_φ este definit prin relaţia

$$(17) \qquad T_\varphi\, u(r, s\,;\,x) = \frac{2}{\pi}\int_0^{\pi/2} \varphi(v(2\sqrt{rs}\sin\theta\,;\,x))\,d\theta$$

pentru orice funcţională liniară continuă φ definită pe $L^2([0, T] \times \Omega)$.

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ON SOME NUMERICAL METHODS FOR GOURSAT PROBLEM

(Summary)

The aim of this Note is to indicate the way for obtaining approximate solutions of Goursat problem for hyperbolic equations of second order, using approximate solutions of an associated Cauchy problem according to [1].

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NUMERICAL SOLUTION OF ORDINARY DIFFERENTIAL EQUATIONS BY THE METHOD OF UNDETERMINED PARAMETERS

BY

K. V. LEUNG and J. VAN KIRK

1. Introduction

A procedure for approximating solutions to first order initial value problems in ordinary differential equations is derived by minimizing an integral over a class of second degree polynomials. A fourth order approximation is obtained. Numerical comparisons with other methods are given.

We will consider the problem

$$y'(x) = f(x, y(x)), \quad y(x_0) = y_0. \tag{1.1}$$

Approximate values, y_n, will be calculated recursively at the grid points, x_n, of a uniform mesh of size h using

$$y_0 = y(x_0), \quad y_{n+1} = p_n(_{n+1}), \tag{1.2}$$

where $p_n(x)$ is some polynomial such that $p_n(x_n) = y(x_n)$.

Loscalzo and Talbot consider the case where $p(x) \equiv p_n(x)$ on $[x_n, x_{n+1}]$ is a spline of degree m and continuity class C^{m-1} imposing the additional condition that $p_n'(x_{n+1}) = f(x_{n+1}, p_n(x_{n+1}))$. They obtained a second order method for $m = 2$, a fourth order method for $m = 3$, but divergent methods for $m > 3$.

Callender [1] relaxed the conditions on $p(x)$ requiring it to be a spline of degree m and class C^1. He imposed the set of conditions

$$p_n'\left(x_n + \frac{jh}{m-1}\right) = f\left(x_n + \frac{jh}{m-1}, \; p_n\left(x_n + \frac{jh}{m-1}\right)\right), \tag{1.3}$$

$$(j = 1, \ldots, m-1)$$

and obtained methods of order m.

For the case of $m = 2$ we have relaxed two additional conditions obtaining a fourth order method. First, the derivative of $p(x)$ is not assu-

med to be continuous, and second, instead of imposing conditions like (1.3) we will minimize an integral,

$$\int_{x_n}^{x_{n+1}} [f(x, p_n(x)) - p_n'(x)]^2 \, dx.$$

2. Method of Undetermined Parameters

Let $p_n(x)$ be the Lagrange interpolation polynomial of degree 2 through (x_n, y_n), $(x_{n+1/2}, y_{n+1/2})$, and (x_{n+1}, y_{n+1}) where $y_{n+1/2}$ and y_{n+1} are parameters to be determined and $x_{n+1/2} = x_n + \frac{h}{2}$. Then

$$p_n(x) = \frac{(x - x_{n+1/2})(x - x_{n+1})}{(x_n - x_{n+1/2})(x_n - x_{n+1})} y_n +$$

$$+ \frac{(x - x_{n+1})(x - x_n)}{(x_{n+1/2} - x_{n+1})(x_{n+1/2} - x_n)} y_{n+1/2} + \frac{(x - x_n)(x - x_{n+1/2})}{(x_{n+1} - x_n)(x_{n+1} - x_{n+1/2})} y_{n+1} =$$

$$= \frac{2}{h^2}(x - x_{n+1/2})(x - x_{n+1}) y_n - \frac{4}{h^2}(x - x_{n+1})(x - x_n) y_{n+1/2} + \tag{2.1}$$

$$+ \frac{2}{h^2}(x - x_n)(x - x_{n+1/2}) y_{n+1}.$$

We want to determine $p_n(x)$ such that $y_{n+1/2}$ and y_{n+1} approximate the true solution to (1.1) at $x_{n+1/2}$ and x_{n+1} respectively. To accomplish this, we will minimize

$$I(y_{n+1/2}, y_{n+1}) = \int_{x_n}^{x_{n+1}} [f(x, p_n(x)) - p_n'(x)]^2 dx. \tag{2.2}$$

Doing this we obtain

$$\frac{1}{2}\frac{\partial I}{\partial y_{n+1/2}} = \int_{x_n}^{x_{n+1}} [f(x, p_n(x)) - p_n'(x)] \left[f_y(x, p_n(x)) \frac{\partial p_n}{\partial y_{n+1/2}} - \right.$$

$$\left. - \frac{\partial p_n'}{\partial y_{n+1/2}} \right] dx = 0, \tag{2.3}$$

$$\frac{1}{2}\frac{\partial I}{\partial y_{n+1}} \int_{x_n}^{x_{n+1}} [f(x, p_n(x)) - p_n'(x)] \left[f_y(x, p_n(x)) \frac{\partial p_n}{\partial y_{n+1}} - \frac{\partial p_n'}{\partial y_{n+1}} \right] dx = 0.$$

Using Simpson's rule to evaluate these integrals, we obtain the following system of non-linear algebraic equations for $y_{n+1/2}$ and y_{n+1}:

$$
\begin{aligned}
0 = & \left[f(x_n, y_n) + \frac{3}{h} y_n - \frac{4}{h} y_{n+1/2} + \frac{1}{h} y_{n+1} \right] \left[-\frac{4}{h} \right] + \\
& + 4 \left[f(x_{n+1/2}, y_{n+1/2}) + \frac{1}{h} y_n - \frac{1}{h} y_{n+1} \right] [f_y(x_{n+1/2}, y_{n+1/2})] + \\
& + \left[f(x_{n+1}, y_{n+1}) - \frac{1}{h} y_n + \frac{4}{h} y_{n+1/2} - \frac{3}{h} y_{n+1} \right] \left[\frac{4}{h} \right]
\end{aligned} \tag{2.4}
$$

and

$$
\begin{aligned}
0 = & \left[f(x_n, y_n) + \frac{3}{h} y_n - \frac{4}{h} y_{n+1/2} + \frac{1}{h} y_{n+1} \right] \left[\frac{1}{h} \right] + \\
& + 4 \left[f(x_{n+1/2}, y_{n+1/2}) + \frac{1}{h} y_n - \frac{1}{h} y_{n+1} \right] \left[-\frac{1}{h} \right] + \\
& + \left[f(x_{n+1}, y_{n+1}) - \frac{1}{h} y_n + \frac{4}{h} y_{n+1/2} - \frac{3}{h} y_{n+1} \right] \left[f_y(x_{n+1}, y_{n+1}) - \frac{3}{h} \right]
\end{aligned} \tag{2.5}
$$

or rewriting (2.4) and (2.5) we get

$$
\begin{cases}
-8y_{n+1/2} + 4y_{n+1} = \Phi_1(y_{n+1/2}, y_{n+1}), \\
16\, y_{n+1/2} - 14\, y_{n+1} = \Phi_2(y_{n+1/2}, y_{n+1}),
\end{cases} \tag{2.6}
$$

where

$$
\begin{aligned}
\Phi_1(y_{n+1/2}, y_{n+1}) = & -4\, y_n + h(f(x_{n+1}, y_{n+1}) - f(x_n, y_n)) + \\
& + hf_y(x_{n+1/2}, y_{n+1/2})(y_n - y_{n+1}) + h^2 f_y(x_{n+1/2}, y_{n+1/2}) f(x_{n+1/2}, y_{n+1/2})
\end{aligned} \tag{2.7}
$$

and

$$
\begin{aligned}
\Phi_2(y_{n+1/2}, y_{n+1}) = & 2y_n + h(f(x_n, y_n) - 4f(x_{n+1/2}, y_{n+1/2}) - \\
& - 3f(x_{n+1}, y_{n+1})) + hf_y(x_{n+1}, y_{n+1})(4y_{n+1/2} - y_n - 3y_{n+1}) + \\
& + h^2 f_y(x_{n+1}, y_{n+1}) f(x_{n+1}, y_{n+1}).
\end{aligned} \tag{2.8}
$$

Then (2.6) can be used to solve iteratively for $y_{n+1/2}$ and y_{n+1} as follows:

$$
\begin{cases}
-8y_{n+1/2}^{(k+1)} + 4y_{n+1}^{(k+1)} = \Phi_1(y_{n+1/2}^{(k)}, y_{n+1}^{(k)}), \\
16\, y_{n+1/2}^{(k+1)2} - 14\, y_{n+1}^{(k+1)} = \Phi_2(y_{n+1/2}^{(k)}, y_{n+1}^{(k)}).
\end{cases} \tag{2.9}
$$

It will be shown later that if the initial guess $(y_{n+1/2}^{(0)}, y_{n+1}^{(0)})$ is made wisely, one iteration will give a good estimate of the true solution.

3. Truncation Error

In (2.9), we can solve for $y_{n+1}^{(k+1)}$ explicitly. We obtain

$$y_{n+1}^{(k+1)} = -\frac{1}{6}\left[2\Phi_1(y_{n+1/2}^{(k)}, y_{n+1}^{(k)}) + \Phi_2(y_{n+1/2}^{(k)}, y_{n+1}^{(k)})\right]. \tag{3.1}$$

Assuming iteration to the limits, (3.1) becomes

$$y_{n+1} = -\frac{1}{6}\left[2\Phi_1(y_{n+1/2}, y_{n+1}) + \Phi_2(y_{n+1/2}, y_{n+1})\right]. \tag{3.2}$$

The truncation error, e_{n+1}, is given by $e_{n+1} = y_{n+1} - y(x_{n+1})$ where $y(x_{n+1})$ represents the true solution. Thus,

$$\begin{aligned} e_{n+1} &= -\frac{1}{6}\left[2\Phi_1(y_{n+1/2}, y_{n+1}) + \Phi_2(y_{n+1/2}, y_{n+1})\right] - y(x_{n+1}) = \\ &= h^5 \frac{1}{2880} f^{\mathrm{IV}} - \frac{1}{576} f_y f^{\mathrm{III}} - \frac{1}{144}(f_{yx} + f_{yy} f) f^{\mathrm{II}}) + O(h^6). \end{aligned} \tag{3.3}$$

Therefore $e_{n+1} = O(h^5)$ so the method is of order 4.

It is of interest to compare the truncation error of this method with that of the classical Runge-Kutta From [3], we have that this error is

$$\begin{aligned} \tilde{e}_{n+1} &= \frac{1}{2880} f^{\mathrm{IV}} - \frac{1}{576} f_y f^{\mathrm{III}} + \frac{1}{288}(f_y^2 - f_{xy} - f_{yy} f) f^{\mathrm{II}} + \\ &+ \frac{1}{192}(2 f_{xy} f_y + 3 f_{yy} f_y f - 2 f_y^3 + f_{yy} f_x) f^{\mathrm{I}}. \end{aligned} \tag{3.4}$$

We observe that the coefficients of f^{IV} and f^{III} are identical in both methods and that the coefficient of f^{I} in the method of undetermined parameters is 0. Therefore if f^{I} is the dominant derivative of f, then this method should be more accurate than the classical Runge-Kutta. In other cases, the two methods should theoretically have comparable errors, but in practice the error in the method of undetermined parameters will be somewhat greater since it is not feasible to iterate to the limit.

4. Numerical Examples

For the first iteration in each step, $f(x_n, y_n)$, $f(x_{n+1/2}, y_{n+1/2})$, $f(x_{n+1}, y_{n+1})$, $f_y(x_{n+1/2}, y_{n+1/2})$, and $f_y(x_{n+1}, y_{n+1})$ must be evaluated. In subsequent iterations, $f(x_n, y_n)$ need not be re-evaluated. However, if $f(x, y) = \sum_{i=1}^{n} a_i(x) b_i(y)$ then the $a_i(x)$ need only be evaluated twice in each iteration (observe that the final value of one step can be used as the initial value in the next step). Therefore, if only one iteration is required and if the $a_i(x)$

are complicated functions and $b_i(y)$ simple functions, then this method has an execution time competitive with standard fourth order techniques.

A computer program was written which used an Euler step to determine $y_{1/2}^{(0)}$ and $y_1^{(0)}$. Three iterations were then executed to determine $y_{1/2}$ and y_1. After the first step, the points y_{n-1}, $y_{n-1/2}$, and y_n and the points y_{n-1}, $y_{n-1/2}$, y_n, and $y_{n+1/2}^{(0)}$ were used in an Adams Bashforth predictor formula to give $y_{n+1/2}^{(0)}$ and $y_{n+1}^{(0)}$ respectively and only one iteration was used.

This method was then compared to the classical Runge-Kutta formula and the fourth order Adams Moulton corrector with third order Adams Bashforth predictor. The calculations were carried out on an IBM 360, model 65, in double precision arithmetic.

Two examples are given. In the first, f^{I} is the dominant derivative and in the second, it is not. For each example, a table is given comparing the total error and execution time of each of these methods.

Example 1.

$$y' = (x^{-1/3} + x^{-1/5} + x^{-1/5} + x^{-1/6})/y^2, \quad y(5) = 10,$$

integrated to $x = 45$.

Table 1

h	New error $\times 10^{-12}$	New exec. time (sec.)	RK error $\times 10^{-12}$	RK exec. time (sec.)	AM error $\times 10^{-12}$	AM exec. time (sec.)
1	17,170	.199	42,330	.278	1,693,000	.171
.5	840	.392	2,677	.557	144,080	.334
.25	43	.777	168	1,118	10,690	.658
.125	2	1.546	10	2,235	731	1,306

Example 2.

$$y' = -(\tan x)\, y + x \sin 2x, \quad y(0) = 1,$$

integrated to $x = 1$.

Table 2

h	New error $\times 10^{-8}$	New exec. time (sec.)	RK error $\times 10^{-8}$	RK exec. time (sec.)	AM error $\times 10^{-8}$	AM exec. time (sec.)
.25	− 67.020	.011	− 20,020	.008	− 494,400	.008
.125	− 4,459	.020	− 1,304	.016	− 28,516	.015
.0626	−281	.040	− 81	.034	− 1,602	.028
.03125	− 17	.076	− 5	.067	− 94	.055

As expected, in the first example, the method of undetermined parameters has the least error. The execution time is faster than that of Runge-Kutta and only somewhat slower than that of Adams Moulton. In the second example, the errors of our method are about three times of that of Runge-Kutta and its execution time is slightly longer, but it again compares favourably with Adams Moulton.

5. Conclusions

Several extensions of this method are now being considered. One is extending this method to higher order polynomials $p_n(x)$ of degree m by using the Lagrange polynomial through the points (x_j, y_j), $x_j = x_n + \frac{jh}{m}$ and $j = 0, 1, \ldots, m$. As before (2.2) would be minimized with respect to the parameters $y_1, \ldots, y_m$.

Also being considered is using the Lagrange polynomial through (x_j, y_j), $x_j = x_0 + jh$ and $j = n - M, \ldots n + N$ where $y_{n-M}, \ldots, y_n$ are the previously computed approximate values and $y_{n+1}, \ldots, y_{n+N}$ are parameters to be determined. In this case,

$$\int_{x_{n-M}}^{x_{n-N}} [f(x, p(x)) - p'(x)]^2 \, dx$$

would be minimized with respect to $y_{n+1}, \ldots, y_{n+N}$. Finally, the extension to higher order equations and systems of equations is under consideration.

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SOLUŢIE NUMERICĂ A ECUAŢIILOR DIFERENŢIALE ORDINARE PRIN METODA COEFICIENŢILOR NEDETERMINAŢI

(Rezumat)

Se obţine soluţia cu aproximaţie de ordinul patru a problemei de valori iniţiale privitoare la ecuaţii diferenţiale ordinare, prin minimizarea unei integrale peste o clasă de polinoame de gradul al doilea.

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СУЩЕСТВОВАНИЕ И ЕДИНСТВЕННОСТЬ РЕШЕНИЙ ОДНОГО КЛАССА ФУНКЦИОНАЛЬНО-РАЗНОСТНЫХ УРАВНЕНИЙ

М. М. КОНСТАНТИНОВ

В настоящей работе найдены достаточные условия существования и единственности решений начальной задачи

$$(1)\qquad \begin{aligned} x(t) &= f(t,\ x(t),\ x(\tau_0)), \quad t > 0, \\ x(t) &= \varphi(t), \quad t \leqslant 0, \end{aligned}$$

где $x=(x_1,\dots,x_n)$, $f=(f_1,\dots,f_n)$, $\varphi=(\varphi_1,\dots,\varphi_n)$. Запаздывающий аргумент τ_0 определен при помощи рекуррентных зависимостей [1]

$$\tau_k = \tau_k(t,\ x(t),\ x(\tau_{k+1}))\ ;\quad k=0,\dots,m-1 \geqslant 0\ ;\ \tau_m = \tau_m(t,\ x(t)).$$

Дальше приняты следующие обозначения : R^n — n - мерное пространство над числовом поле R ; $|\ .\ |$ — норма в R^n ; C — пространство непрерывных функций $x : I_T \to R^n$, $I_x = (-\infty,\ T]$, $T > 0$; $\|x\| = \sup\{\,|x(t)|\ :\ t \in I_T\}$ — норма в C ; $F(t)$ — неотрицательная, ограниченная на интервале $I = (0,\ T]$ скалярная функция ;

$$\Omega = \{\xi : |\,\xi\,| \leqslant \sup\{\,F(t) : t \in I\}\},\quad \Omega_0 = \Omega \cup \{\bigcup_{t=-\infty}^{0} \{\varphi(t)\}\},$$

подмножества в R^n ; $X(\beta)$ — множество всех функций $x \in C$, удовлетворяющих условию Липшица с постоянной β и таких, что $x(t)=\varphi(t)$ при $t \in I_0$ и $|\,x(t)\,| \leqslant F(t)\,|$ при $t \in I$.

Предположим, что выполнена следующая группа условий А) :

А1. Функции $f(t,\ \xi,\ \eta)$ и $\tau_s\,(t,\ \xi,\ \eta)$, $s=0,\dots,m$, определены при $(t,\ \xi,\ \eta) \in G = I \times \omega = I \times \Omega \times \Omega_0$ и удовлетворяют там условиям Липшица

$$|f(t,\ \xi,\ \eta) - f(\bar{t},\ \bar{\xi},\ \bar{\eta})| \leqslant L\,|\,t-\bar{t}\,| + M\,|\,\xi-\bar{\xi}\,| + N\,|\,\eta-\bar{\eta}\,|,$$

$$|\,\tau_s\,(t,\ \xi,\ \eta) - \tau_s(\bar{t},\ \bar{\xi},\ \bar{\eta})\,| \leqslant \lambda\,|\,t-\bar{t}\,| + \mu\,|\,\xi-\bar{\xi}\,| + \nu\,|\,\eta-\bar{\eta}\,|.$$

А2. Выполнены условия

$$\inf\{t - \tau_s(t,\ \xi,\ \eta) : (t,\ \xi,\ \eta,\ s) \in G \times \overline{0,\ m}\} \geqslant 0,$$

$$\sup\{\,|f(t,\ \xi,\ \eta)\,| : (\xi,\ \eta) \in \omega\} \leqslant F(t).$$

A3. Функция $\varphi(t)$ удовлетворяет условию Липшица

$$|\varphi(t)-\varphi(\bar{t})| \leqslant \beta |t-\bar{t}|; \quad t, \bar{t} \in I_0.$$

A4. Выполнено условие согласования $\varphi(0)=f(0, \varphi(0), \varphi(\tau_0^0))$, где $\tau_k^0=\tau_h(0, \varphi(0), \varphi(\tau_{k+1}^0))$; $k=0, \ldots, m-1$; $\tau_m^0=\tau_m(0, \varphi(0))$.

Теорема 1. *Пусть выполнены условия A). Пусть кроме того* $M<1$ *и* $M+L\nu+N\lambda+2\sqrt{LN(\mu+\lambda\nu)} \leqslant 1$.

Тогда если $\beta \leqslant \beta_2$:

$$\beta_2=\frac{1}{2a}(b+\sqrt{b^2-4aL}); \; a=N\mu+\nu(1-M), \; b=1+L\nu-M-N\lambda,$$

то начальная задача (1) *имеет хотя бы одно решение* $x \in X(\tilde{\beta})$, *где*

$$\tilde{\beta}=\max\{\beta, \beta_1\}, \quad \beta_1=\frac{b}{a}-\beta_2. \tag{2}$$

Доказательство. Отметим, что в силу условий теоремы числа β_1, β_2 существуют и притом $\nu\beta_2<1$.

Пусть оператор Π действует в $X(\tilde{\beta})$ по формуле $\Pi x(t)=f(t, x(t), x(\tau_0))$ при $t \in I$ и $\Pi x(t)=\varphi(t)$ при $t \in I_0$. Покажем, что непрерывный оператор Π преобразует множество $X(\tilde{\beta})$ в себя. Действительно, функция $\Pi x(t)$ удовлетворяет условиям $\Pi x(t)=\varphi(t)$, $t \in I_0$ и $|\Pi x(t)| \leqslant F(t)$, $t \in I$ при $x \in X(\tilde{\beta})$. Оценим число $\Delta(t, \bar{t})=|\Pi x(t)-\Pi x(\bar{t})|$, где без ограничения общности примем $t, \bar{t} \in I$ Имеем

$$\begin{aligned}\Delta(t, \bar{t}) &\leqslant L|t-\bar{t}|+M|x(t)-x(\bar{t})|+N|x(\tau_0)-x(\bar{\tau}_0)| \leqslant \\ &\leqslant (L+M\tilde{\beta})|t-\bar{t}|+N\tilde{\beta}|\tau_0-\bar{\tau}_0|,\end{aligned} \tag{3}$$

где $\bar{\tau}_k=\tau_k(\bar{t}, x(\bar{t}), x(\bar{\tau}_{k+1}))$; $k=0, \ldots, m-1$; $\bar{\tau}_m=\tau_m(\bar{t}, x(\bar{t}))$. Учитывая, что

$$|\tau_k-\bar{\tau}_k| \leqslant (\lambda+\mu\tilde{\beta})|t-\bar{t}|+\nu\tilde{\beta}|\tau_{k+1}-\bar{\tau}_{k+1}|;$$

$$k=0, \ldots, m-1; \quad |\tau_m-\bar{\tau}_m| \leqslant (\lambda+\mu\tilde{\beta})|t-\bar{t}|,$$

получаем

$$|\tau_0-\bar{\tau}_0| \leqslant \frac{\lambda+\mu\tilde{\beta}}{1-\nu\tilde{\beta}}|t-\bar{t}|. \tag{4}$$

Подставляя (4) в (3) и используя (2) приходим к оценке

$$\Delta(t, \bar{t}) \leqslant \left(L+M\tilde{\beta}+N\tilde{\beta}\frac{\lambda+\mu\tilde{\beta}}{1-\nu\tilde{\beta}}\right)|t-\bar{t}|=$$

$$=\left(\tilde{\beta}+\frac{a\tilde{\beta}^2-b\tilde{\beta}+L}{1-\nu\tilde{\beta}}\right)|t-\bar{t}| \leqslant \tilde{\beta}|t-\bar{t}|.$$

Итак мы показали, что $\Pi X(\tilde{\beta}) \subset X(\tilde{\beta})$. Так как множество $X(\tilde{\beta})$ выпукло, замкнуто и относительно компактно, то в силу принципа Шаудера существует хотя бы одна инвариантная точка $x \in X(\tilde{\beta})$ оператора П. Теорема доказана.

Теорема 2. *Пусть выполнены условия теоремы 1. Пусть кроме того*

$$q = M + N\left(1 + \frac{\mu + \nu}{1 - \nu\tilde{\beta}}\tilde{\beta}\right) < 1.$$

Тогда начальная задача (1) *имеет единственное решение* $x \in X(\tilde{\beta})$.

Доказательство. Пусть $x, x^* \in X(\tilde{\beta})$. Тогда

$$|\Pi x(t) - \Pi x^*(t)| \leqslant M\,|x(t) - x^*(t)| + N\,|x(\tau_0) - x^*(\tau_0^*)| \leqslant$$
$$\leqslant (M + N)\|x - x^*\| + N\tilde{\beta}\,|\tau_0 - \tau_0^*|,$$

где $\tau_k^* = \tau_k(t, x^*(t), x^*(\tau_{k+1}^*))$; $k = 0, \ldots, m-1$; $\tau_m^* = \tau_m(t, x^*t))$.

Имея ввиду, что

$$|\tau_k - \tau_k^*| \leqslant (\mu + \nu)\|x - x^*\| + \nu\tilde{\beta}\,|\tau_{k+1} - \tau_{k+1}^*|\,;$$
$$k = 0, \ldots, m-1\,;\ |\tau_m - \tau_m^*| \leqslant \mu\,\|x - x^*\|,$$

получаем

$$|\tau_0 - \tau_0^*| \leqslant \frac{\mu + \nu}{1 - \nu\tilde{\beta}}\,\|x - x^*\|,$$

т. е.,

$$\|\Pi x - \Pi x^*\| \leqslant \left(M + N + N\tilde{\beta}\,\frac{\mu + \nu}{1 - \nu\tilde{\beta}}\right)\|x - x^*\| = q\|x - x^*\|.$$

Следовательно П является оператором сжатия на множестве $X(\tilde{\beta})$ и согласно принципу Банаха существует единственная инвариантная точка $x \in X(\tilde{\beta})$ оператора П. Теорема доказана.

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EXISTENŢA ŞI UNICITATEA SOLUŢIEI UNEI CLASE DE ECUAŢII FUNCŢIONALE CU DIFERENŢE

(Rezumat)

Se demonstrează existenţa şi unicitatea soluţiei ecuaţiei funcţionale vectoriale cu diferenţe $x(t) = f(t, x(t), x(\tau_0))$, $t > 0$ cu condiţiile iniţiale $x(t) = \varphi(t)$, $t \leqslant 0$.

Argumentul transformat se determină prin relaţiile $\tau_k = \tau_k(t, x(t), x(\tau_{k+1}))$, $k=0,\ldots, m-1$, $\tau_m = \tau_m(t, x(t))$.

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REPREZENTAREA PLANĂ A OBIECTELOR PE BAZA CORESPONDENŢEI AFINE ÎN DIEDRUL DE REFERINŢĂ VARIABIL[1]

DE

I. RUSU

1. Introducere

Urmînd ordinea de idei din [3], [4], autorul consideră proiecţia ortogonală a corpului (obiectului) dat pe planul $[H]$ al reprezentării. Se proiectează pe planul de proiecţie $[P]$ un punct caracteristic al obiectului, proiecţia acestuia găsindu-se intersectînd proiectanta corespunzătoare, cu planul $[P]$. În acest fel corespondenţa afină între proiecţia ortogonală dată şi cea cilindrică oblică din planul ce trebuie construit este determinată. Imaginea (proiecţia) din planul de proiecţie $[P]$ se construieşte pe baza acestei corespondenţe.

2. Construcţia proiecţiei afine tridirecţionale a unui corp geometric

Fie un hexaedru regulat (cub) de muchie dată, care are faţa $ABCE$ situată pe planul reprezentării $[H]$ (fig. 1). Se cere construcţia proiecţiei oblice a acestui poliedru pe un plan $[P]$ determinat de dreapta $(\Delta) = [H] \cap [P]$ şi de un punct $F(f; 2,5)$. Direcţia de proiectare este dreapta $D(d\,;\, D_0)$ determinată de punctele $A(a\,;\, 0)$ şi $S(s\,;\, 2,5)$.

Pe planul reprezentării cubul se proiectează ortogonal după pătratul $m \equiv a\,;\; n \equiv b\,;\; k \equiv c\,;\; l \equiv e$. Ca şi în alte cazuri se consideră planul $Q \perp (\Delta)$, care trece prin punctul F. Urmele sale se confundă $(Qh) = (Qp)$ şi trec prin $f(2,5)$ fiind perpendiculare pe axa de omologie $\Delta \equiv \delta$. Rabătînd planul Q pe planul reprezentării se rabate de fapt urma sa (Qp). Pentru aceasta se duce prin f perpendiculara pe $(Qh) \equiv (Qp)$ şi se măsoară $fF = 2,\ 5$ unităţi.

Punctul F rabătut şi cu punctul $U = (Qh) \cap (\Delta)$, determină urma (Q_1p) rabătută pe planul reprezentării $[H]$.

Se consideră apoi prin vîrful $A(a\,;\, 0)$ şi prin punctul $S(s\,;\, 2,5)$ direcţia D de proiectare. Intersecţia dintre această dreaptă şi planul de proiecţie $[P]$ este punctul $A_1(a_1\,;\, a_1A_1)$. Proiectat ortogonal pe planul $[H]$ se obţine proiecţia a_1 şi rabătut (rotit) în planul $[H]$ în jurul punctului R_δ; cercul de rotaţie fiind situat în planul vertical $[R]$, se obţine punctul A_{10}, una din cele două soluţii ale rabaterii indicate.

Punctele a_1 şi A_{10} pot fi considerate afine ale vîrfului $A(a\,;\, 0)$ şi deci omologia afină este determinată.

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[1]) Prezentată la sesiunea ştiinţifică a Institutului de construcţii din Bucureşti, din 24—25 aprilie 1969.

Intersecţia $A_1(a_1\,;\,Z_A) = (D) \cap [P]$ se determină în felul următor. Se consideră planul vertical $[R]$ dus prin direcţia de proiectare $D(A\,;\,S)$. Planul $[R]$ se intersectează cu planul $[P]$ după urma (Rp) care este determinată de punctul $R_\delta = (Rh) \cap (\Delta)$ şi de un punct F_1 de pe aceeaşi orizontală a planului $[P]$, dar la intersecţia cu (Rp).

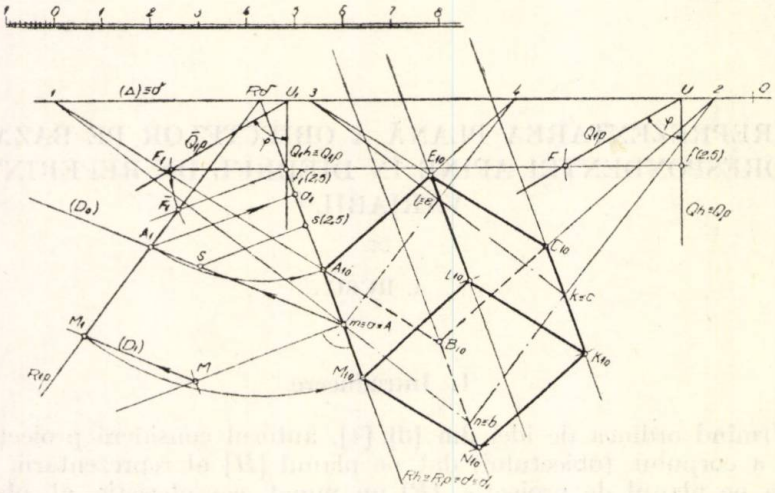

Fig. 1

Deci $F_1 = (FF_1) \cap (Rp)$ şi prin urmare: $fF_1 = f_1F_1 = f_1F_1 = Z_F = Z_{F_1} = 2{,}5$ unităţi.

Ca şi prin punctul F, se consideră prin punctul menţionat F_1 planul $[Q_1] \perp (\Delta)$, deci: $[Q_1] \parallel [Q]$.

Din această cauză urmele de acelaşi nume sînt paralele, fapt ce se desprinde cu uşurinţă din epură (fig. 1).

Cu centrul în proiecţia f_1 şi de rază f_1F_1 se duce un arc de cerc, care intersectează perpendiculara la $(Rh) \equiv (Rp)$ dusă tot prin f_1 în F_1. Acest punct de fapt este tot punctul F_1 rabătut în planul $[H]$ în jurul axei de rabatere (Rh).

Se consideră apoi dreapta (R_1p) determinată, aşa cum s-a arătat în punctele R şi F_1, precum şi dreapta (D_0) care este direcţia de proiectare (D) rabătută în planul reprezentării $[H]$, determinată de punctele A şi S. Prin intersecţia acestor drepte se obţine punctul A_1, pentru care s-a construit proiecţia a_1 ortogonală pe planul $[H]$ şi poziţia rabătută A_{10}. Acestea sînt puncte afine ale vîrfului A.

Prelungind laturile pătratului, ca proiecţie ortogonală pe planul reprezentării a cubului considerat, acestea intersectează axa de omologie afină în punctele *1*, *2*, *3* şi *4*. Aceste drepte pot fi considerate ca intersecţii dintre planele feţelor laterale ale poliedrului dat şi planul $[H]$ al reprezentării. Dreapta $1A_{10}$ se intersectează cu proiectanta vîrfului $B(b\,;\,0)$, ca paralelă cu prima proiectantă $D(A\,;\,S)$, în punctul B_{10}. Apoi dreapta $2B_{10}$ intersectează proiectanta vîrfului $C(c\,;\,0)$ în C_{10} şi în mod analog se află şi vîrful E_{10}.

Dreptele $(1A_{10})$; $(2B_{10})$; $(3\,C_{10})$ şi $(4E_{10})$ sînt intersecţiile dintre planele aceloraşi feţe laterale ale cubului dat cu planul de proiecţie $[P]$ deci sînt afinele dreptelor $(1\,a)$; $(2\,b)$; $(3\,c)$ şi $(4\,e)$. Paralelogramul $A_{10}\,B_{10}\,C_{10}\,E_{10}$ este proiecţia oblică pe planul $[P]$ a feţei $ABCE$, deci este figura afină a acestui pătrat.

Cum planul vertical $[R]$ conţine muchia laterală AM a cubului dat, odată cu acest plan se rabate în planul reprezentării şi această muchie. De aceea prin proiecţia $m \equiv a$ se duce perpendiculara la (Rh) pe care se măsoară segmentul aM egal cu muchia cubului.

Apoi prin punctul M rabătut se duce proiectanta acestui vîrf $D_1 \equiv D_0$, pentru că proiecţia utilizată este cilindrică oblică, care se intersectează cu urma (R_1p) în punctul M_1. Se rabat în planul reprezentării punctele proiecţie A_1 şi M_1, în care scop se duc arcele de cerc

cu centrul în R_δ şi de rază $R_\delta A_1$ şi $R_\delta M_1$. Aceste arce intersectează urma $(Rh) \equiv (d) \equiv (d_1)$ în M_{10}. Deci $A_{10} M_{10}$ este proiecţia cilindrică oblică a muchiei laterale AM pe planul $[P]$.

Cum muchiile cubului sînt egale între ele, rezultă că toate muchiile laterale paralele şi egale între ele se vor proiecta, în sistemul menţionat, după segmente de dreaptă paralele şi egale cu proiecţia găsită $A_{10} M_{10}$. De aceea, pe proiectantele proiecţii ale vîrfurilor $B_{10} C_{10}$ şi E_{10} se măsoară începînd din aceste vîrfuri segmentele $B_{10} N_{10} = C_{10} K_{10} = E_{10} L_{10} = A_{10} M_{10}$.

În felul acesta s-au găsit proiecţiile vîrfurilor cubului şi ţinîndu-se seama de vizibilitate, proiecţia poliedrului se poate construi unind prin linii drepte proiecţiile vîrfurilor, aşa cum se observă în epură. Se trasează mai întîi paralelogramul $M_{10}N_{10}K_{10}L_{10}$ şi după aceasta proiecţiile muchiilor laterale arătate mai sus.

Pentru stabilirea vizibilităţii s-a considerat că sînt văzute punctele şi muchiile cubului considerat care sînt mai apropiate de planul de proiecţie $[P]$, exact aşa cum este reflectată în oglindă imaginea unui obiect, cu singura deosebire că normalele la planul oglinzii sînt înlocuite, în cazul de faţă, cu proiectantele (D) care fac un unghi diferit de cel drept cu planul $[P]$.

Observaţie. În cazul construcţiei proiecţiilor cilindrice oblice a corpurilor cilindro-conice, se va reprezenta mai întîi proiecţia cercului director, care aşa cum s-a arătat este elipsă, considerată curbă afină a acestuia. Proiecţia se va completa considerîndu-se generatoarele de contur aparent paralele cu o direcţie dată în cazul cilindrului şi concurente într-un punct dat, în cazul conului.

3. Aplicaţii ale proiecţiei afine a corpurilor geometrice

Se consideră un element de construcţie (arhitectură) de exemplu o boltă. Detaliul menţionat este alcătuit din două coloane de forma unor prisme pătrate drepte ale căror feţe interioare sînt racordate la partea superioară printr-o pînză semicilindrică care este bolta propriu zisă. Toată această alcătuire de corpuri geometrice, prisme şi cilindrul bolţii, este proiectată ortogonal pe planul reprezentării $[H]$ (fig. 2).

Obiectul considerat trebuie proiectat oblic pe planul $[P]$ de proiecţie, care este determinat, aşa cum s-a menţionat mereu de dreapta $(\Delta) \equiv (\delta)$ ca intersecţie dintre planele corespondente $[H]$ şi $[P]$ şi de punctul $M(m\,;\,3)$.

Dreapta (Δ) este şi în acest caz axă de omologie afină. Ca şi în cazurile precedente se consideră planul $[Q]$ dus prin punctul M şi perpendicular pe $[H]$. Se observă că proiecţia acestui corp pe planul $[H]$ este un dreptunghi, care are unul din vîrfuri notat cu $s \equiv c \equiv n \equiv \equiv a \equiv A$. Aceasta reprezintă muchia verticală corespunzătoare vîrfului $A(a\,;\,0)$ pe care se găsesc următoarele puncte caracteristice:

a) Punctul $N(n\,;\,nN)$, în care muchia menţionată este intersectată în planul de nivel, de cotă $nN = Z_N$, al naşterii bolţii considerate.

b) Punctul $C(c\,;\,cC)$ reprezintă intersecţia aceleiaşi muchii cu planul de nivel, de cotă $cC = Z_C$, tangent la cheia bolţii.

c) Punctul $S(s\,;\,sS)$ este intersecţia muchiei verticale considerate cu planul de nivel de cotă $sS = Z_S$, care limitează înălţimea obiectului dat.

Se consideră proiectanta vîrfului A determinată deci de acest punct şi de punctul $L(l\,;\,3{,}5)$. Pentru a o intersecta cu planul $[P]$, se consideră planul vertical $[R]$ prin această proiectantă şi prin muchia (AS). Rezultă urma (Rp) de intersecţie dintre planul $[R]$ şi planul $[P]$. Se rabate planul $[R]$ împreună cu dreptele conţinute, fapt care conduce la obţinerea intersecţiei $A_1 = (D_0) \cap (R_1p)$. Construcţia urmei (R_1p), care este urma (Rp) rabătută pe planul reprezentării $[H]$, se face analog cu cazurile precedente, adică se consideră punctul M_1 din planul $[P]$, dar care aparţine şi planului ajutător, $[R]$ şi care se găseşte pe aceeaşi orizontală a planului $[P]$ ca şi punctul M dat. Deci: $M_1 = (MM_1) \cap [R]$ şi deci: $Z_M = Z_{M_1} = 3$ unităţi.

Se construiesc urmele planului $[Q] \supset M$ şi fiindcă $[Q] \perp (\Delta)$, rezultă că $[Q_1] \parallel [Q]$ de unde urmele: $(Q_1h) \parallel (Qh)$ şi $(Q_1p) \parallel (Qp)$.

Se duce prin $m_1(3)$ perpendiculara la (Rh) pe care se măsoară $m_1M_1 = mM = 3$ unităţi. Punctele R şi M_1 determină urma (R_1p) adusă prin rabatere în planul $[H]$. Cu ajutorul punctelor A şi L a fost rabătută odată cu planul $[R]$ şi direcţia de proiectare în planul reprezentării $[H]$, care este notată cu (D_0). Aşa cum s-a arătat, din intersecţia dreptelor (R_1p) şi (D_0) rezultă proiecţia A_1, care este rotită în jurul punctului R_δ, prin trasarea arcului de cerc cu centrul în R_δ şi de rază $R_\delta A_1$, ce intersectează urma (Rh) în punctul A_{20} care reprezintă una din cele două soluţii ale rotaţiei.

Odată cu planul $[R]$ se rabate şi muchia verticală (AS). Pentru aceasta se duce prin a perpendiculara la (Rh) pe care se figurează punctele caracteristice obiectului şi care au fost enumerate la început. Se va ţine seama în acest sens de următoarele: $Z_N = \overline{AN}$, $Z_C = \overline{AC}$ şi $Z_S = \overline{AS}$.

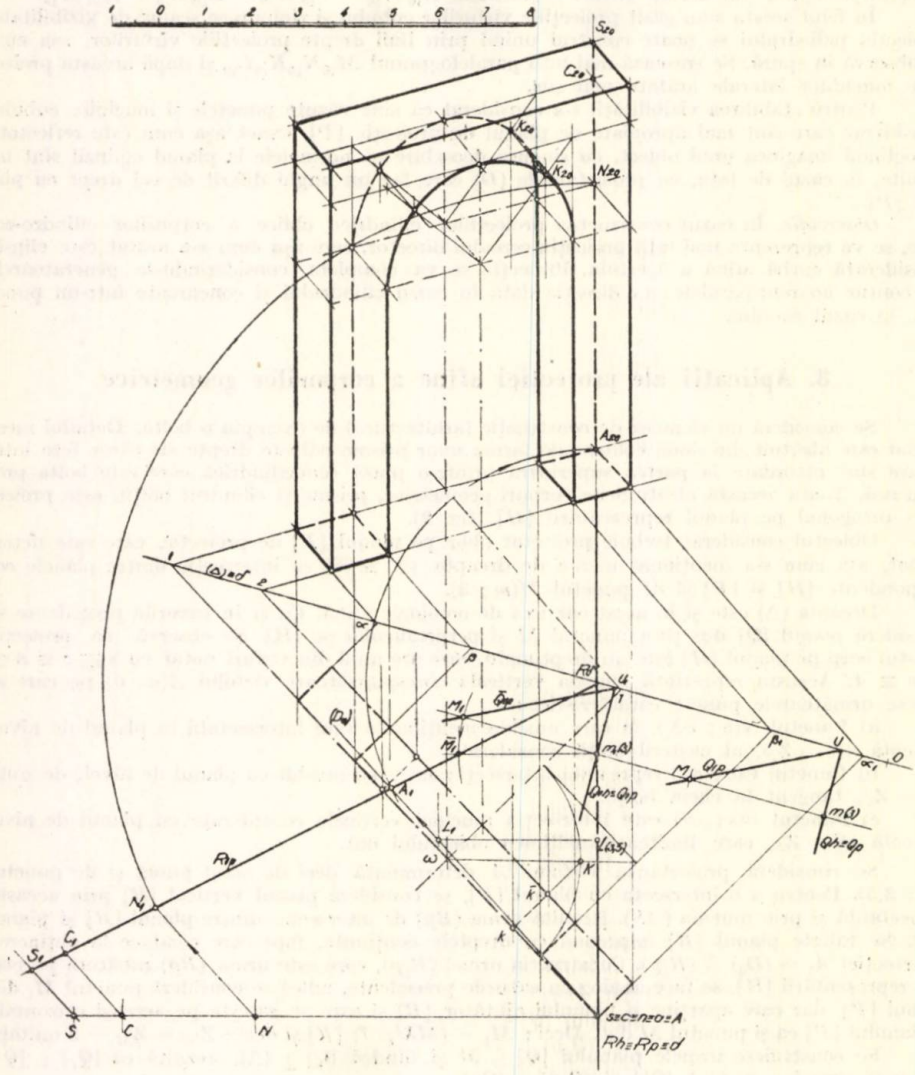

Fig. 2

Apoi din punctele rabătute N, C şi S se duc paralele la (D_0) care întîlnesc urma (R_1p) în N_1, C_1 şi S_1, ce reprezintă proiecţiile pe planul $[P]$ ale punctelor corespunzătoare de pe muchia verticală A şi care rotite (rabătute) în acelaşi sens şi în jurul aceluiaşi centru R conduc la găsirea pe (Rh) a soluţiilor N_{20}, C_{20} şi S_{20}. Prelungind laturile proiecţiei din planul $[H]$, acestea intersectează axa de omologie afină în punctele 1; 2; α; β; I; β_1 şi α_1.

Se uneşte punctul A_{20} prin drepte cu punctele *1* şi *2*. Din punctul *2* se duce o paralelă la dreapta afină $1\,A_{20}$. Din β_1 ; *1* ; β şi α se duc paralele la dreapta afină $\alpha_1 A_{20}$. S-a format astfel reţeaua afină cu ajutorul căreia se trasează proiecţiile afine ale bazelor celor două coloane. Prin vîrfurile acestor poligoane afine se duc proiecţiile muchiilor laterale ca paralele cu proiecţia $(A_{20}\,S_{20})$.

Prin N_{20} se duc paralele la laturile paralelogramului bazei obiectului, care intersectează muchiile laterale ale coloanelor bolţii în punctele de naştere sau limită ale semielipselor directoare ca proiecţii corespunzătoare ale bolţii.

De asemenea prin proiecţia C_{20} se duc drepte paralele la aceleaşi laturi ale bazei bolţii, ce se intersectează cu axele de simetrie ale acelor bolţi în punctele cheie. Se consideră şi proiecţiile K_{20} de pe arcele de elipsă care rezultă din intersecţia acestora cu diagonala semipătratului circumscris. În acest scop pe proiecţia din planul $[H]$ s-a figurat sfertul de cerc ω şi sfertul de pătrat circumscris. Diagonala ωk îl intersectează (este vorba de cercul ω) în punctul K. Acest punct ridicat conduce la găsirea proiecţiilor k şi $\bar{k}$ cărura li se găsesc şi punctele simetrice în raport cu axa de revoluţie a bolţii.

Ducînd paralele la proiecţia $(A_{20}\,S_{20})$ din punctele k găsite, se obţin la intersecţia cu diagonalele proiecţie punctele k_{20}, care împreună cu proiecţiile găsite ale punctelor de naştere şi cheie dau posibilitatea trasării semielipselor directoare ale proiecţiei cilindrice oblice a bolţii considerate.

Se ţine seama de vizibilitate considerînd că punctele, muchiile sau generatoarele obiectului dat care sînt mai apropiate de planul de proiecţie $[P]$ sînt văzute, excepţie făcînd conturul marginal al proiecţiei care este văzut în orice caz.

4. Concluzii

1. Obţinerea imaginii tridirecţionale a unui obiect prezintă avantajul important că aceasta precizează clar forma obiectului sau detaliului, precum şi poziţia sa într-un ansamblu. Cunoscîndu-se direcţia de proiectare se poate afla, aşa cum s-a indicat, perechea de puncte afine, care împreună cu axa de afinitate şi proiecţia de pe planul reprezentării $[H]$ determină corespondenţa planelor $[H]$ şi $[P]$ ale diedrului de referinţă, ceea ce face posibilă construcţia proiecţiei obiectului pe planul $[P]$.

2. Lucrarea oferă o metodă de construcţie grafică în scopul reprezentării directe a proiecţiilor tridirecţionale a obiectelor din domeniul tehnicii.

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EBENE DARSTELLUNG DER KÖRPER AUF GRUND DER AFFINEN KORRESPONDENZ IM VARIABLEN REFERENZDIEDER

(Zusammenfassung)

Die Konstruktion der schiefen parallelen Projektion der Körper wurde unter Annahme des Dieders als Referenzsystem realisiert. Auf der Darstellungsebene $[H]$ ergibt sich der Körper durch seine senkrechte Projektion und auf der Projektionsebene, $[P]$, erhält man dessen Dreirichtungsbild (Projektion), auf Grund der affinen Korrespondenz welche sich zwischen den obengenannten Projektionen bildet.

BULETINUL INSTITUTULUI POLITEHNIC IAŞI
Secţia I
MATEMATICĂ
MECANICĂ TEORETICĂ. FIZICĂ

C.Z. 518.4.

APLICAŢII ALE METODEI COORDONATELOR VECTORIALE PLAN-AXIALE ÎN CINETOSTATICA SOLIDULUI GHIDATE CU PUNCTE

I. DETERMINAREA PARAMETRILOR CINEMATICI DE ORDINUL ÎNTÎI AI MIŞCĂRII SOLIDULUI

DE

C. HUIU, EUGEN BUŢURCĂ, TATIANA TACU şi V. LUIS

Pentru a ilustra modul de rezolvare grafică, bazat pe metoda prezentată în [3]...[7], a problemelor de cinematica şi dinamica solidului rigid, se va efectua, în cele ce urmează, studiul cinetostatic al unui rigid cu un punct cu mişcare comandată şi cu alte două puncte ghidate, unul pe o suprafaţă şi celălalt pe o curbă spaţială, toate legăturile fiind considerate staţionare şi cu frecare neglijabilă.

În această primă Notă se aplică metoda grafică a coordonatelor vectoriale plan-axiale la determinarea parametrilor cinematici de ordinul întîi ai solidului, necesari rezolvării problemei de cinetostatică propuse, urmînd ca în notele următoare să se prezinte determinarea grafică, prin aceeaşi metodă, a parametrilor cinematici de ordinul doi, precum şi a reacţiunilor celor două legături aplicate rigidului.

Se consideră deci cunoscute, într-un moment oarecare al mişcării, următoarele date:

a) Componentele plan-axiale ale vectorului de poziţie, ale vitezei şi ale acceleraţiei punctului cu mişcare comandată, considerat drept pol, Q, invariabil legat de solid:

$$\bar{r}_Q = \bar{r}_{Q\Pi} + \bar{r}_{Q\Delta}, \quad \bar{v}_Q = \bar{v}_{Q\Pi} + v_{Q\Delta}, \quad \bar{a}_Q = \bar{a}_{Q\Pi} + \bar{a}_{Q\Delta}. \tag{1}$$

b) Componentele plan-axiale ale vectorului de poziţie al punctului B_2 al solidului situat pe curba spaţială de ghidare (Γ_2), ale versorului tangenţial $\bar{\tau}_2$ al curbei (Γ_2) în B_2, componentele plan-axiale ale versorului $\bar{\nu}_2$ al normalei principale în B_2 la aceeaşi curbă, versorul $\bar{\nu}_2^*$ al normalei principale al curbei proiectate pe planul reprezentării, adică:

$$\begin{cases} \bar{r}_2 = (\bar{r}_{Q\Pi} + \bar{r}'_{2\Pi}) + (\bar{r}_{Q\Delta} + \bar{r}'_{2\Delta}), \quad \bar{\alpha}_2 = \cos\psi_2^\tau \bar{t}_2 + \sin\psi_2^\tau \bar{u}, \\ \bar{\nu}_2 = \cos\psi_2^\nu \bar{\nu}_2^* + \sin\psi_2^\nu \bar{u}, \quad \bar{n}_2 = \bar{n}_2, \end{cases} \tag{2}$$

precum şi centrele de curbură $K_2(K_{2\Pi}, K_{2\Delta})$ al curbei (Γ_2) şi $C_2(C_{2\Pi}, C_{2\Delta})$ al proiecţiei ortogonale a curbei (Γ_2) pe planul reprezentării. În relaţiile (2), ψ_2^τ este unghiul format cu planul reprezentării de tangenta în B_2 la curba (Γ_2), ψ_2^ν unghiul format cu acelaşi plan de normala

principală în B_2 la curba (Γ_2), $\bar{t}$ versorul tangenţial la curba proiectată, iar $\bar{u}$ versorul axei reprezentării.

c) Componentele plan-axiale ale vectorului de poziţie al punctului B_3 de sprijin al solidului pe suprafaţa (Σ):

$$\bar{r}_3 = (\bar{r}_{Q\Pi} + \bar{r}'_{3\Pi}) + (\bar{r}_{Q\Delta} + \bar{r}'_{3\Delta}). \tag{3}$$

d) Componentele $\bar{r}'_{3\Pi}$ şi $\bar{r}'_{3\Delta}$ fiind determinate cu ajutorul condiţiilor de rigiditate pentru sistemul de puncte Q, B_2 şi B_3 ($\overline{QB_3}$ = const.; $\overline{B_2B_3}$ = const.) şi din condiţia ca punctul B_3 să se găsească pe suprafaţa (Σ) —, precum şi componentele plan-axiale ale versorului $\bar{\nu}_3$ al normalei în B_3 la suprafaţa (Σ):

$$\bar{\nu}_3 = \cos\psi_3^{\nu}\,\bar{\nu}_3^{*} + \sin\psi_3^{\nu}\,\bar{u}, \tag{4}$$

$\bar{\nu}_3^{*}$ fiind versorul proiecţiei ortogonale a acestei normale pe planul reprezentării, iar ψ_3^{ν} — unghiul format de normală cu planul reprezentării. De asemenea, se mai consideră cunoscută poziţia $K(K_\Pi, K_\Delta)$ a centrului de curbură al curbei de intersecţie a suprafeţei (Σ) cu planul determinat de normala la suprafaţă (Σ) şi de direcţia vitezei punctului B_2.

Pentru rezolvarea problemei propuse se va determina mai întîi distribuţia vitezelor, ceea ce revine în fond, la găsirea componentelor plan-axiale ale vitezelor punctelor B_2 şi B_3:

$$\bar{v}_2 = \bar{v}_Q + \bar{\omega} + \bar{r}'_2 \quad \text{şi} \quad \bar{v}_3 = \bar{v}_a + \bar{\omega} \times \bar{r}_3, \tag{5, 6}$$

determinare ce se va face în baza sistemelor de relaţii:

$$\bar{v}_{\sigma\Delta}(\bar{r}'_{\sigma\Delta} - \bar{r}'_{1\Delta}) + \bar{v}_{\sigma\Pi}(\bar{r}'_{\sigma\Pi} - \bar{r}'_{1\Pi}) = \bar{v}_{1\Delta}(\bar{r}'_{\sigma\Delta} - \bar{r}'_{1\Delta}) + \bar{v}_{1\Pi}(\bar{r}'_{\sigma\Pi} - \bar{r}'_{1\Pi}) \quad (\sigma = 2, 3) \tag{7}$$

şi:

$$|\bar{v}_{\sigma\Delta}| = |\bar{v}_{\sigma\Pi}|\,\mathrm{tg}\,\psi_\sigma^{\nu}, \quad (\sigma = 2, 3) \tag{8}$$

care particularizate la condiţiile acestei probleme, ($\bar{r}'_{1\Delta} = 0$; $\bar{r}'_{1\Pi} = 0$; $\bar{v}_{1\Delta} = \bar{v}_{Q\Delta}$; $\bar{v}_{2\Pi} = \bar{v}_{Q\Pi}$), devin:

$$\bar{v}_{2\Delta}\bar{r}'_{2\Delta} + \bar{v}_{2\Pi}\bar{r}'_{2\Pi} = \bar{v}_{Q\Delta}\bar{r}'_{2\Delta} + \bar{v}_{Q\Pi}\bar{r}'_{2\Pi}, \quad |\bar{v}_{2\Delta}| = |\bar{v}_{2\Pi}|\,\mathrm{tg}\,\psi_2^{\tau}, \tag{9}$$

pentru punctul B_2 şi:

$$\left\{\begin{aligned} &\bar{v}_{3\Delta}(\bar{r}'_{3\Delta} - \bar{r}'_{2\Delta}) + \bar{v}_{3\Pi}(\bar{r}'_{3\Pi} - \bar{r}'_{2\Pi}) = \bar{v}_{2\Delta}(\bar{r}'_{3\Delta} - \bar{r}'_{2\Delta}) + \bar{v}_{2\Pi}(\bar{r}'_{3\Pi} - \bar{r}'_{2\Pi}), \\ &\bar{v}_{3\Delta}\bar{r}'_{3\Delta} + \bar{v}_{3\Pi}\bar{r}'_{3\Pi} = \bar{v}_{Q\Delta}\bar{r}'_{3\Delta} + \bar{v}_{Q\Pi}\bar{r}'_{3\Pi}, \end{aligned}\right. \tag{10}$$

pentru punctul B_3 de pe suprafaţa (Σ), prima dintre relaţiile (10) rezultînd prin considerarea punctului B_2 drept punct legat de solid.

Relaţiile (9) vor permite să se determine mărimile $|\bar{v}_{2\Delta}|$ şi $|\bar{v}_{2\pi}|$ acestea avînd expresiile:

$$|\bar{v}_{2\Pi}| = \left|\frac{\bar{v}_{Q\Delta}\bar{r}'_{2\Delta} + \bar{v}_{Q\pi}\bar{r}'_{2\Pi}}{h_2\,\mathrm{tg}\,\psi_2^{\tau} + |\bar{r}'_{2\Pi}|\cos(\bar{r}'_{2\Pi}, \bar{t}_2)}\right|, \tag{11}$$

$$(12)\qquad |\bar{v}_{2\Delta}| = \left| \frac{\bar{v}_{Q\Delta}\,\bar{r}'_{2\Delta} + \bar{v}_{Q\Pi}\,\bar{r}'_{2\Pi}}{h_2 \operatorname{tg}\psi_2^{\tau} + |\bar{r}'_{2\Pi}| \cos(\bar{r}'_{2\Pi}, \bar{t}_2)} \right| \operatorname{tg}\psi_2^{\tau},$$

cu $h'_2 = \mathrm{pr}_{\bar{v}_{2\Delta}}\,\bar{r}'_{2\Delta}$, aceste mărimi putînd fi ușor de găsit grafic cu ajutorul vectorilor reprezentativi corespunzători [3].

Considerînd în condiția de ortogonalitate dintre vectorii $\bar{v}_3$ și $\bar{\nu}_3$:

$$(13)\qquad \mathrm{pr}_{\bar{u}}\,\bar{v}_{3\Delta} \sin\psi_3^{\nu} + v_{3\Pi}\,\bar{\nu}_3^{*} \cos\psi_3^{\nu} = 0$$

versorul $\bar{\nu}_3^{*}$ descompus, în planul reprezentării, odată după versorii $\bar{\rho}_{30} = \mathrm{vers}\,\bar{r}'_{3\Pi}$ și $\bar{n}_{30}$ normal la $\bar{r}'_{3\Pi}$ și a doua oară după versorii $\bar{\rho}_{32} = \mathrm{vers}\,(\bar{r}'_{3\Pi} - \bar{r}'_{2\Pi})$ și $\bar{n}_{32}$ normal la $\bar{\rho}_{32}$ se poate scrie conform fig. 1:

$$(14)\qquad \bar{\nu}_3^{*} = \cos\chi_{30}\,\bar{\rho}_{30} + \sin\chi_{30}\,\bar{n}_{30}, \qquad (15)\qquad \bar{\nu}_3^{*} = \cos\chi_{32}\,\bar{\rho}_{32} + \sin\chi_{32}\,\bar{n}_{32}.$$

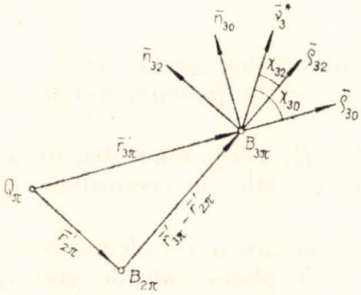

Fig. 1

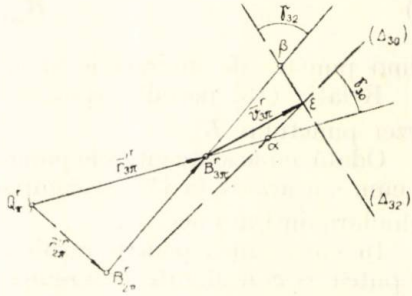

Fig. 2

În baza acestor expresii, rezultă din (13) relațiile:

$$(16)\qquad \mathrm{pr}_{\bar{u}}\,\bar{v}_{3\Delta} = v_{3\Delta} = -\operatorname{cotg}\psi_3^{\nu} \cos\chi_{30}\,\mathrm{pr}_{\bar{\rho}_{30}}\,\bar{v}_{3\Pi} - \operatorname{cotg}\psi_3^{\nu} \sin\chi_{30}\,\mathrm{pr}_{\bar{n}_{30}}\,\bar{v}_{3\Pi},$$

$$(17)\qquad \mathrm{pr}_{\bar{u}}\,\bar{v}_{3\Delta} = v_{3\Delta} = -\operatorname{cotg}\psi_3^{\nu} \cos\chi_{32}\,\mathrm{pr}_{\bar{\rho}_{32}}\,\bar{v}_{3\Pi} - \operatorname{cotg}\psi_3^{\nu} \sin\chi_{32}\,\mathrm{pr}_{\bar{n}_{32}}\,\bar{v}_{3\Pi}.$$

Folosindu-se aceste ultime două relații, ecuațiile (10) pot fi aduse la formele:

$$(18)\qquad \begin{cases} \mathrm{pr}_{\bar{\rho}_{30}}\,\bar{v}_{3\Pi} = \dfrac{\mathrm{pr}_{\bar{\rho}_{30}}\,\bar{v}_{Q\Pi} + \dfrac{h_3}{|\bar{r}'_{3\Pi}|}\,v_{Q\Delta}}{1 - \dfrac{h_3}{|\bar{r}'_{3\Pi}|}\operatorname{cotg}\psi_3^{\nu}\cos\chi_{30}} + \dfrac{\dfrac{h_3}{|\bar{r}'_{\Pi}|}\operatorname{cotg}\psi_3^{\nu}\sin\chi_{30}}{1 - \dfrac{h_3}{|\bar{r}'_{3\Pi}|}\operatorname{cotg}\psi_3^{\nu}\cos\chi_{30}}\,\mathrm{pr}_{\bar{n}_{30}}\,\bar{v}_{3\Pi}, \\[2ex] \mathrm{pr}_{\bar{\rho}_{32}}\,\bar{v}_{3\Pi} = \dfrac{\mathrm{pr}_{\bar{\rho}_{32}}\,\bar{v}_{2\Pi} + \dfrac{(h_3 - h_2)}{|\bar{r}'_{3\Pi} - \bar{r}'_{2\Pi}|}\,v_{Q\Delta}}{1 - \dfrac{h_3 - h_2}{|\bar{r}'_{3\Pi} - \bar{r}'_{2\Pi}|}\operatorname{cotg}\psi_3^{\nu}\cos\chi_{32}} + \dfrac{\dfrac{h_3 - h_2}{|\bar{r}'_{3\Pi} - \bar{r}'_{2\Pi}|}\operatorname{cotg}\psi_3^{\nu}\cos\chi_{32}}{1 - \dfrac{h_3 - h_2}{|\bar{r}'_{3\Pi} - \bar{r}'_{2\Pi}|}\operatorname{cotg}\psi_3^{\nu}\cos\chi_{32}}\,\mathrm{pr}_{\bar{n}_{32}}\,\bar{v}_{3\Pi} \end{cases}$$

conducînd la ecuaţii între vectorii reprezentativi, de forma :

$$(19) \qquad \mathrm{pr}_{QB_3}\, \bar{v}^r_{3\Pi} = b^r_{30}\, \mathrm{tg}\, \gamma_{30}\, \mathrm{pr}_{\bar{n}_{30}}\, \bar{v}^r_{3\Pi}\,, \quad \mathrm{pr}_{B_2B_3}\, \bar{v}^r_{3\Pi} = b^r_{32}\, \mathrm{tg}\, \gamma_{32}\, \mathrm{pr}_{\bar{n}_{32}}\, \bar{v}^r_{3\Pi}\,,$$

cu segmentele b^r_{30} şi b^r_{32} şi unghiurile γ_{30} şi γ_{32} cunoscute (ca şi direcţiile versorilor $\bar{n}_{30}$ şi $\bar{n}_{32}$).

Ultimele ecuaţii vor permite să se determine vectorul $\bar{v}^r_{3\Pi}$ cu ajutorul a două drepte loc geometric al extremităţilor acestui vector.

Prima dreaptă (Δ_{30}) (fig. 2) trece prin punctul α, $(\overline{B^r_{3\Pi}\alpha} = b^r_{30})$ şi formează unghiul γ_{30} cu segmentul $Q^r_{30}B^r_{3\Pi}$. A doua dreaptă loc geometric (Δ_{32}) trece prin punctul β, $(\overline{B^r_{3\Pi}\beta} = b^r_r)$ şi formează unghiul γ_{32} cu segmentul $B^r_{2\Pi}B^r_{3\Pi}$.

Este uşor de verificat că în baza relaţiilor (19) componenta plană a vitezei punctului B_3 va fi dată de vectorul :

$$(20) \qquad \overrightarrow{B^r_{3\Pi}\varepsilon} = \bar{v}^r_{3\Pi}\,,$$

ε fiind punctul de intersecţie al celor două drepte loc geometric.

Relaţia (13) permite apoi să se găsească şi componenta axială $\bar{v}_{3\Delta}$ a vitezei punctului B_3.

Odată cunoscute vitezele punctelor Q, B_2 şi B_3 vor putea fi determinate, aşa cum s-a arătat în [7], şi componentele plan-axiale ale vectorului viteză unghiulară instantanee, $\bar{\omega}$.

În consecinţă, pentru rezolvarea în continuare a problemei propuse vor putea fi considerate cunoscute componentele plan-axiale ale vectorilor reprezentativi ai parametrilor cinematici de ordinul întîi ai solidului [6], adică :

$$(21) \qquad \bar{v}^r_{Q_\Pi} = \frac{\bar{v}_{Q\Pi}}{\sigma_v}\,, \quad \bar{v}^r_{Q_\Delta} = \frac{\bar{v}_{Q\Delta}}{\sigma_v}\,; \qquad \bar{\omega}^r_\Pi = \frac{\bar{\omega}_\Pi}{\sigma_\omega}\,, \quad \bar{\omega}^r_\Delta = \frac{\bar{\omega}_\Delta}{\sigma_\omega}\,.$$

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UNE APPLICATION DE LA MÉTHODE DES COORDONNÉES VECTORIELLES PLANE-AXIELLES DANS LA CINÉTOSTATIQUE DU RIGIDE, AVEC DES POINTS GUIDÉS

I. Détermination des paramètres cinématiques de premier ordre

(Résumé)

Étant donné un rigid, dont un point $B_1 \equiv Q$ a un mouvement commandé, caractérisé par les grandeurs cinématiques (1), et les points B_2 et B_3 sont guidés, le premier sur une courbe spacielle (Γ_2), le dernier sur une surface (Σ), avec toutes les caractéristiques géométriques connues, on présente, dans cette première Note, la détermination graphique des paramètres cinématiques de premier ordre du rigide, qui seront utilisés ensuite dans la résolution du problème cinétostatique.

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C. D. 518.4

FORCES PERCUTANTES DANS LES POINTS IRRÉGULIERS ET EXTENSION DES RELATIONS DE FRENET

PAR

ILIE N. CONSTANTINESCU, EMIL TOCACI et TRAIAN NOAGHI

1. Soit un point matériel M en mouvement sur une courbe (C) continue, pour laquelle les points $P_i (i = 1, 2, ..., n)$ sont des points irréguliers à double tangente (ou de retour). Celle-ci peut être la trajectoire d'une particule matérielle libre, ou soumise aux liaisons. Quelle que soit la situation, dans les points P_i, la particule M subit des chocs, puisque le vecteur vitesse a une variation instantanée finie, comme conséquence de la particularité des points P_i.

Évidemment, dans le cas des particules libres, les forces percutantes ne peuvent être qu'extérieures et, en cas des liaisons, elles peuvent être des forces de liaison. Indifféremment de la nature des forces extérieures, si (C) este une courbe matérialisée à laquelle est attaché M, en P_i on enregistrera des forces percutantes de liaison.

Soit α_{i1}, α_{i2} les angles faits par les tangentes à (C) en P_i et $\bar{\tau}_{i1}$ et $\bar{\tau}_{i2}$ les verseurs des deux tangentes en P_i.

2. En ce qui suit, nous nous proposons d'établir l'expression des forces percutantes, ou des accélérations, dans les points irréguliers.

À cet effet, il est nécessaire d'établir l'expression de la dérivée du verseur $\bar{\tau}$, par rapport au temps, en ces points.

On observe que la percussion subie par la particule M, est la même dans le point P_i, indifféremment si le mouvement est fait sur la courbe (C) ou s'il est effectué sur les tangentes à (C) en P_i. La trajectoire formée par les demidroites $-\bar{\tau}_1$ et $\bar{\tau}_2$, sera nommée courbe associée (Γ).

Puisque ce que nous avons établi pour un seul point P_1 maintient sa valabilité pour tous les autres, nous traiterons le cas d'un seul point P_1, en renonçant à l'indice, pur simplifier les expressions.

En même temps la remarque ci-dessus permet de considérer l'expression de $\bar{\tau}$ dans un voisinage suffisament petit du P_1, elle étant donnée par la relation

$$\bar{\tau} = \bar{\tau}_1 H(t_0 - t) + \bar{\tau}_2 H(t - t_0), \tag{1}$$

où $H(u)$ est la fonction de Heaviside et t_0 le moment où le mobile passe par P_1

$$H(u) = \begin{cases} 0 & \text{pour} \quad u < 0, \\ 1 & \text{pour} \quad u > 0. \end{cases} \tag{2}$$

En dérivant (1) et tenant compte du fait que sur (Γ) $\bar{\tau}_1$ et $\bar{\tau}_2$ sont des verseurs constants, il résulte

$$\dot{\bar{\tau}} = (\bar{\tau}_2 - \bar{\tau}_1)\,\delta(t - t_0), \tag{3}$$

où δ est la distribution de Dirac. La propriété bienconnue

$$\delta(t - t_0) = \delta(t_0 - t) \tag{4}$$

a été utilisée pour obtenir l'égalité (3).

Puisque la vitesse de la particule est

$$\bar{v} = v\bar{\tau} \tag{5}$$

son accélération sera :

$$\bar{a} = \dot{\bar{v}}\bar{\tau} + v\dot{\bar{\tau}}, \tag{6}$$

ou de (3) :

$$\bar{a} = \dot{v}\bar{\tau} + v(\bar{\tau}_2 - \bar{\tau}_1)\,\delta(t - t_0) \tag{6'}$$

dans un voisinage du P_1.

D'autre part, si s est l'arc de la courbe (C), dans un voisinage suffisamment petit du P_1, l'expression du verseur $\bar{\tau}$ peut être mise sous la forme :

$$\bar{\tau} = \bar{\tau}_1 H(s_0 - s) + \bar{\tau}_2 H(s - s_0), \tag{7}$$

s_0 étant l'arc correspondant au P_1.

En assimilant le mouvement réel à celui sur la courbe associée et en dérivant (7) par rapport au temps t on arrive à la relation :

$$\dot{\bar{\tau}} = (\bar{\tau}_2 - \bar{\tau}_1)\, v\, \delta(s - s_0), \tag{8}$$

où $v = \dot{s}$ est la grandeur de la vitesse.

En comparant les expressions (3) et (8) du $\dot{\bar{\tau}}$, on obtient :

$$v\delta(s - s_0) = \delta(t - t_0) \tag{9}$$

et de la comparaison du (7) au (1) il résulte :

$$H(s - s_0) = H(t - t_0). \tag{9'}$$

En ces conditions, l'expression de l'accélération dans le voisinage considéré, devient :

$$\bar{a} = \dot{v}\bar{\tau} + (\bar{\tau}_2 - \bar{\tau}_1)\, v^2 \delta(s - s_0). \tag{10}$$

On conclut, que la force dans le point singulier est

$$\bar{F} = m\dot{v}\bar{\tau} + (\bar{\tau}_2 - \bar{\tau}_1)mv^2\,\delta(s - s_0). \tag{11}$$

3. Il est intéressant à observer que l'on peut aussi éteindre les relations de Frenet sur ces dernières catégories de mouvements.

De la sorte, si α est le demi-angle formé par $-\bar{\tau}_1$ et $\bar{\tau}_2$, on peut facilemment établier l'égalité

$$\bar{\tau}_2 - \bar{\tau}_1 = 2\cos\alpha\,\bar{\nu}, \tag{12}$$

$\bar{\nu}$ étant le verseur de la direction de la bissectrice de l'angle formé par $\bar{\tau}_2$ et $-\bar{\tau}_1$.

Par conséquent, les relations (10) et (11) prennent la forme

$$\bar{a} = \dot{v}\bar{\tau} + 2\cos\alpha\, v^2\delta(s - s_0)\,\bar{\nu} \tag{10'}$$

et respectivement :

$$\bar{F} = m\dot{v}\tau + 2m\cos\alpha v^2\,\delta(s - s_0)\bar{\nu}. \tag{11'}$$

D'autre part, la courbe associée (Γ) formée par les semidirections $-\bar{\tau}_1$ et $\bar{\tau}_2$, peut être regardée comme l'élément dégénéré d'une famille de courbes, particulierement d'une famille d'hyperboles ($\mathcal{H}$), qui admet comme assymptotes, la courbe (Γ).

4. Indépendamment de tout cela, nous nous proposons d'établir la courbure de (Γ) (en utilisant la théorie des distributions), et en la considérant comme la limite des hyperboles de la famile ($\mathcal{H}$).

La généralité ne souffre pas si, pour simplifier le calcul, on choisit un système de référence Px_0y_0 avec l'origine dans le point irrégulier P et l'axe Py_0 en coïncidence avec la bissectrice de l'angle formé par les directions $-\bar{\tau}_1$ et $\bar{\tau}_2$.

À remarquer que, si la courbe (C) este une courbe plane, située dans le plan Oxy, celle-ci, évidemment, coïncidera avec le plan Px_0y_0, et si la courbe (C) este une courbe irrégulière dans l'espace $Oxyz$, elle peut être assimilée dans une voisinage suffisamment petit du P, à une courbe plane, dans le plan de la courbe (Γ), c'est-à-dire dans le plan Px_0y_0, ce que signifie que ce qui suit maintient sa valabilité dans toutes les deux situations, car tous les phénomènes mécaniques étudiés sont situés dans le voisinage du point P.

Soit maintenant la famille des courbes du plan Px_0y_0

$$y_0 = \frac{b}{a}\sqrt{x_0^2 + \lambda^2}, \tag{13}$$

où λ est un paramètre. Soit K_λ la courbure d'un élément de cet ensemble d'hyperboles.

Si

$$I_\lambda = \int K_\lambda \, dx_0, \tag{14}$$

on vérifie immédiatement, tenant compte du (13), que

$$\lim_{\lambda \to 0} J_\lambda \Big|_p^q = \begin{cases} 0 & \text{pour } 0 \bar{\in} (p, q), \\ \dfrac{2b}{(a^2 + b^2)^{1/2}} & \text{pour } 0 \in (p, q), \end{cases} \tag{15}$$

aussi bien que l'inégalité

$$\left| I_\lambda \Big|_p^q \right| < \frac{2b}{\sqrt{a^2 + b^2}}, \tag{16}$$

existente pour n'importe quel p, q et λ. Il en résulte :

$$\lim_{\lambda \to 0} K_\lambda = \frac{2b}{\sqrt{a^2 + b^2}} \, \delta(x - x_0). \tag{17}$$

Mais, parce que

$$\frac{a}{b} = \operatorname{tg} \alpha, \tag{18}$$

ce que fait que dans le cas limite ($\lambda \to 0$) les courbes (13) dégénérant en (Γ), on peut écrire :

$$\lim_{\lambda \to 0} K_\lambda = K, \tag{19}$$

en notant par K la courbure de la courbe associée.

En vertu des relations (17), (18) et (19), on obtient :

$$K = 2 \cos \alpha \, \delta(x_0). \tag{20}$$

En observant que cette égalité a été obtenue dans le cas $\lambda \to 0$ et tenant compte du (13) et (19), aussi bien que du fait que dans un voisinage suffisamment petit du P, les courbes (C) et (Γ) coïncident, on peut affirmer que

$$\delta(x - x_0) = \delta(s - s_0) \, \delta(P), \tag{21}$$

$\delta(P)$ étant la distribution de Dirac concentré dans le point P.

Les égalités (21) modifient quelques unes des relations ci-dessus, ainsi la courbure sur (Γ) devient

$$K = 2 \cos \alpha \, \delta(P) \tag{20'}$$

et l'accélération de la particule sur la courbe (C) prend la forme

$$\bar{a} = \dot{v} \bar{\tau} + 2 \cos \alpha \, \delta(P) \, v^2 \bar{\nu}, \tag{10'}$$

où

$$(10'') \qquad \bar{a} = \dot{v}\bar{\tau} + Kv^2\bar{\nu},$$

modifiant aussi l'expression de la force :

$$(10''') \qquad \bar{F} = m\dot{v}\bar{\tau} + mKv^2\bar{\nu}.$$

L'extension des relations de Frenet aux courbes à points irréguliers, facilite, en outre, la détermination des forces percutantes appliquées aux particules matérielles en mouvement sur des courbes contenant des points irréguliers.

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FORŢE PERCUTANTE ÎN PUNCTE NEREGULATE ŞI EXTINDERI ALE RELAŢIILOR LUI FRENET

(Rezumat)

Se generalizează expresiile vitezei şi acceleraţiei în coordonate intrinseci pentru cazul curbelor cu puncte neregulate, obţinînd expresia forţei care acţionează o particulă materială care evoluează pe o traiectorie ce conţine puncte neregulate.

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D.C. 539.3

ON SOME THEOREMS IN A GENERALIZED THEORY OF LINEAR THERMOELASTICITY

BY

ELISABETA RUSU

1. Introduction

The generalized theory of linear thermoelasticity was established by A. E. Green and K. A. Lindsay [1] with the help of an entropy production inequality proposed by E. Green and N. Laws [2]. In [3] D. Ieşan gives some theorems in dynamic theories of continuum mechanics and derives the reciprocity theorems in a form which leads to variational theorems of Gurtin [4] type.

In this paper the reciprocity theorem is derived and a variational principle, using the method given by Ieşan. The reciprocity theorem is derived without using Laplace transform and a variational theorem for initial-boundary value problem is established, for the generalized theory of linear thermoelasticity, extending the results from [1], [3], [5], [6].

2. Preliminary Considerations

Let V be a regular (in sense of Kellog) region of the space occupied by considered medium whose boundary is Σ.

Let u and v be functions defined on $V\times[0, \infty)$ continuous on $[0, \infty)$ in respect to the time t, for each $x \in V$. We denote by $u \ast v$ the convolution of u and v

$$u \ast v = \int_0^t u(x, t-\tau)\, v(x, \tau)\, \mathrm{d}\tau \tag{2.1}$$

and we will use the properties of the convolution.

We will denote by $u \otimes v$ the „convolution scalar product" [3], [7]

$$(u \otimes v) = \int_V u \ast v \, \mathrm{d}x. \tag{2.2}$$

Let us consider the equation

$$Au = f, \tag{2.3}$$

where A is a linear operator. It is known [8] that if A is a positive operator defined on a dense set in a real Hilbert space, then a solution of the equation (2.3) minimizes the functional

$$\mathcal{F}(u) = (Au, u) - 2(u, f) \tag{2.4}$$

and conversely.

If the linear operator A is only simmetric in convolution [3], [7]

$$(Au \otimes v) = (u \otimes Av) \quad \text{for} \quad u, v \in D_A, \tag{2.5}$$

we consider the functional

$$F(u) = (Au \otimes u) - 2(u \otimes f). \tag{2.6}$$

Then

$$\delta F(u) = 0 \quad \text{on} \quad D_A \tag{2.7}$$

if, and only if $u \in D_A$ satisfies the equation (2.3).

3. Basic Equations

The basic equations for the generalized linear theory of thermoelasticity [1], are:

(i) The equations of motion:

$$t_{ij,j} + \rho F_i = \rho \ddot{u}_i. \tag{3.1}$$

(ii) The equation of entropy:

$$\theta_0 \rho \dot{\eta} + q_{i,i} = \rho r. \tag{3.2}$$

(iii) The constitutive equations:

$$\begin{aligned} t_{ij} &= C_{ijkl}\, e_{kl} + a_{ij}(\theta + \alpha\dot{\theta}), \\ \rho\eta &= a + d\theta + h\dot{\theta} - a_{ij}\, e_{ij}, \\ q_i &= -\theta_0 k_{ij}\, \theta_{,j} \end{aligned} \tag{3.3}$$

(we suppose that the initial body has a centre of simmetry at each point a_i and b_i vanishes [1])

(iv) The strain displacement relations:

$$2e_{ij} = u_{i,j} + u_{j,i}. \tag{3.4}$$

The notations are: u_i — components of displacement vector; t_{ij} — components of the stress tensor; e_{ij} — components of the strain tensor; F_i — specific body force; η — the specific entropy; θ — the temperature measured from a constant reference temperature θ_0; q_i — components of heat

flux vector; r — specific heat supplied; C_{ijkl}, a_{ij}, k_{ij}, a, d, h, ρ, α - the characteristics of the material. We have

$$C_{ijkl} = C_{klij} = C_{jikl}, \quad a_{ij} = a_{ji}, \quad k_{ij} = k_{ji}. \tag{3.5}$$

To the system of field equations we adjoin the initial conditions

$$u_i(x, 0) = u_i(x), \quad u_i(x, 0) = v_i(x), \quad \theta(x, 0) = 0, \quad \eta(x, 0) = \eta_0(x) \tag{3.6}$$

and the boundary conditions:

$$\begin{aligned} &u_i = \tilde{u}_i \text{ on } \Sigma_1 \times [0, \infty), \qquad t_i = t_{ij} n_j = \tilde{t}_i \text{ on } \Sigma_2 \times [0, \infty), \\ &\theta = \tilde{\theta} \text{ on } \Sigma_3 \times [0, \infty), \qquad q = q_i n_i = \tilde{q} \text{ on } \Sigma_4 \times [0, \infty), \end{aligned} \tag{3.7}$$

where u_{0i}, v_{0i}, $\rho\eta_0(x)$ (or $\dot{\theta}(0)$), $\tilde{u}_i$, $\tilde{t}_i$, $\tilde{\theta}$, $\tilde{q}$ are prescribed functions, n_i are the components of the unit outward normal to Σ, and Σ_α ($\nu = 1, 2, 3, 4$) denotes subsets of Σ such that

$$\Sigma_1 \cup \Sigma_2 = \Sigma_3 \cup \Sigma_4 = \Sigma, \qquad \Sigma_1 \cap \Sigma_2 = \Sigma_3 \cap \Sigma_4 = \varnothing.$$

We will derive first the reciprocity theorem using the method given in [3], [5], [9], [10].

We will denote by l, g the functions defined on $[0, \infty)$ by:

$$l(t) = 1, \quad g(t) = t. \tag{3.8}$$

Let f_i, W be functions defined on $\overline{V}[0, \infty)$ by

$$\rho f_i = g \ast \rho F_i + \rho(t v_{0i} + u_{0i}), \qquad W = \frac{1}{\theta_0} l \ast \rho r + \rho\eta_0 - a. \tag{3.9}$$

Theorem 3.1. *The functions u_i, t_{ij}, q_i, satisfy the equations* (3.1) (3.2) *and the initial conditions* (3.6) *if, and only if*

$$g \ast t_{ij,j} + \rho f_i = \rho u_i, \tag{3.10}$$

$$(\rho\eta - a) + \frac{1}{\theta_0} l \ast q_{i\,i} = W. \tag{3.11}$$

Let us consider the body subject to two different systems of thermoelastics loadings:

$$L^\nu = (\rho F_i^{(\nu)}, \rho r^{(\nu)}, \tilde{u}_i^{(\nu)}, \tilde{t}_i^{(\nu)}, \tilde{\theta}^{(\nu)}, \tilde{q}^{(\nu)}, u_{0i}^{(\nu)}, v_{0i}^{(\nu)}, \eta_0^{(\nu)}), \quad (\nu = 1, 2) \tag{3.12}$$

and the two corresponding states $C^{(\nu)} = \{u_i^{(\nu)}, e_{ij}^{(\nu)}, t_{ij}^{(\nu)}, \theta^{(\nu)}, q_i^{(\nu)}, \eta^{(\nu)}\}$.

We introduce the notations:

$$\begin{aligned} &\rho f_i^{(\nu)} = g \ast \rho F_i^{(\nu)} + \rho(t v_{0i}^{(\nu)} + u_{0i}^{(\nu)}), \quad W^{(\nu)} = \frac{1}{\theta_0} l \ast \rho r^{(\nu)} + \rho\eta_0^{(\nu)} - a, \\ &t_i^{(\nu)} = t_{ij}^{(\nu)} n_j, \quad q^{(\nu)} = q_i^{(\nu)} n_i, \quad (\nu = 1, 2). \end{aligned} \tag{3.13}$$

Theorem 3.2. *If a thermoelastic solid is subjected to two different system of thermoelastic loadings $L^{(\nu)}$, then between the corresponding thermoelastic states $C^{(\nu)}$ there is a following reciprocity relation:*

$$
\begin{aligned}
&\int_V \rho f_i^{(1)} \ast u_i^{(2)}\, dx - \int_V (g \ast W^{(1)} + l \ast \alpha W^{(1)}) \ast \theta^{(2)}\, dx + \\
&+ \int_\Sigma g \ast \left[t_i^{(1)} \ast u_i^{(2)} + \frac{1}{\theta_0} (l \ast q^{(1)} + \alpha q^{(1)}) \ast \theta^{(2)} \right] d\Sigma = \\
&= \int_V \rho f_i^{(2)} \ast u_i^{(1)}\, dx - \int_V (g \ast W^{(2)} + l \ast \alpha W^{(2)}) \ast \theta^{(1)}\, dx + \\
&+ \int_\Sigma g \ast \left[t_i^{(2)} \ast u_i^{(1)} + \frac{1}{\theta_0} (l \ast q^{(2)} + \alpha q^{(2)}) \ast \theta^{(1)} \right] d\Sigma.
\end{aligned}
\tag{3.14}
$$

Proof. On the basis of the properties of convolution, (3.5), from (3.3) we get

$$
[t_{ij}^{(1)} - a_{ij} (\theta^{(1)} + \alpha \dot{\theta}^{(1)})] \ast e_{ij}^{(2)} = [t_{ij}^{(2)} - a_{ij} (\theta^{(2)} + \alpha \dot{\theta}^{(2)})] \ast e_{ij}^{(1)},
$$

$$
[a - \rho \eta^{(1)} + h \dot{\theta}^{(1)} - a_{ij}\, e_{ii}^{(1)}] \ast \theta^{(2)} = [a - \rho \eta^{2} + h \dot{\theta}^{(2)} - a_{ij}\, e_{ij}^{(2)}] \ast \theta^{(1)},
$$

$$
[\alpha(a - \rho \eta^{(1)}) + \alpha d \theta^{(1)} - \alpha a_{ij}\, e_{ij}^{(1)}] \ast \dot{\theta}^{(2)} = [\alpha(a - \rho \eta^{(2)}) + \alpha d \theta^{(2)} - \alpha a_{ij}\, e_{ii}^{(2)}] \ast \dot{\theta}^{(1)}.
$$

Adding up these relations we obtain

$$
\begin{aligned}
&t_{ij}^{(1)} \ast e_{ij}^{(2)} + (a - \rho \eta^{(1)}) \ast (\theta^{(2)} + \alpha \dot{\theta}^{(2)}) + h \dot{\theta}^{(1)} \ast \theta^{(2)} + \alpha d \theta^{(1)} \ast \dot{\theta}^{(2)} = \\
&= t_{ij}^{(2)} \ast e_{ij}^{(1)} + (a - \rho \eta^{(2)}) \ast (\theta^{(1)} + \alpha \dot{\theta}^{(1)}) + h \dot{\theta}^{(2)} \ast \theta^{(1)} + \alpha d \theta^{(2)} \ast \dot{\theta}^{(1)}.
\end{aligned}
\tag{3.15}
$$

If we introduce the notations

$$
\begin{aligned}
I_{\mu\nu} = \int_V g \ast [& t_{ij}^{(\mu)} \ast e_{ij}^{(\nu)} + (a - \rho \eta^{(\mu)}) \ast (\theta^{(\nu)} + \alpha \dot{\theta}^{(\nu)}) + \\
& + h \dot{\theta}_{(\mu)} \ast \theta^{(\nu)} + \alpha d \theta^{(\mu)} \ast \dot{\theta}^{(\nu)}]\, dx
\end{aligned}
\tag{3.16}
$$

from (3.15) we have

$$
I_{12} = I_{21}. \tag{3.17}
$$

Using (3.4), (3.10), (3.11), the divergence theorem, the property of convolution $g \ast u = l \ast (l \ast u)$ and the initial condition $\theta(x, 0) = 0$, we obtain:

$$I_{\mu\nu} = \int_{\Sigma} g * t_{ij}^{(\mu)} n_j * u_i^{(\nu)} \, d\Sigma + \int_{\Sigma} \left[g * \left(l * \frac{1}{\theta_0} q_i^{(\mu)} n_i + \frac{\alpha}{\theta_0} q_i^{(\mu)} n_i \right) * \theta^{(\nu)} \right] d\Sigma +$$

$$(3.18) \quad + \int_V [\rho f_i^{(\mu)} * u_i^{(\nu)} - (g * w^{(\mu)} + l * \alpha w^{(\mu)}) * \theta^{(\nu)}] \, dx - \int_V \rho u_i^{(\mu)} * u_i^{(\nu)} \, dx +$$

$$+ \int_V g * (l * k_{ij} \theta_{,j}^{(\mu)} + \alpha k_{ij} \theta_{,j}^{(\mu)}) * \theta_{,i}^{(\nu)} \, dx + \int_V l * (h + \alpha d) \theta^{(\mu)} * \dot{\theta}^{(\nu)} \, dx.$$

From (3.5), (3.17) and (3.18) we obtain (3.14). The proof is complete.

4. Variational Principle

Now, we will derive the variational principle. In this end we will write the reciprocity theorem in another form. We introduce the vectors

$$(4.1) \qquad U = (u_1, u_2, u_3, \theta), \quad F = (\rho f_1, \rho f_2, \rho f_3, \rho f_4)$$

where ρf_i are $(3.9)_1$ and $\rho f_4 = -g * W - l * \alpha W$.

From (3.3), (3.10) and (3.11) can be written in the form of a vector equation.

Theorem 4.1. *The functions* u_i, θ, *satisfy the equations* (3.10) *and* (3.11), *if and only if*

$$(4.2) \qquad AU = F,$$

where

$$A_i U = -g * [C_{ijkl} u_{k,l} + a_{ij}(\theta + \alpha\dot{\theta})]_{,j} + \rho u_i,$$

$$(4.3) \qquad A_4 U = g * [l * (k_{ij} \theta_{,j})_{,i} + \alpha (k_{ij} \theta_{,j})_{,i}] + g * (a_{ij} u_{i,j} -$$

$$- d\theta - h\dot{\theta}) + l * (\alpha a_{ij} u_{i,j} - \alpha d\theta - \alpha h\dot{\theta}).$$

It follows that the vector U is a solution of the mixed problem (3.1)—(3.7), if and only if

$$(4.4) \qquad AU = F,$$

$$(4.5) \qquad u_i = \tilde{u}_i \quad \text{on } \Sigma_1 \times [0, \infty), \qquad \theta = \tilde{\theta} \quad \text{on } \Sigma_3 \times [0, \infty),$$

$$(4.6) \qquad t_i = \tilde{t}_i \quad \text{on } \Sigma_2 \times [0, \infty), \qquad q = \tilde{q} \quad \text{on } \Sigma_4 \times [0, \infty).$$

The reciprocity relation (3.14) can be written in the form:

$$(4.7) \qquad \int_V (AU * V - U * AV) \, dx = \int_{\Sigma} g * (U * \tau V - V * \tau U) \, d\Sigma,$$

where

$$
(4.8)\qquad
\begin{aligned}
\tau U &= \left(t_1^{(1)},\ t_2^{(1)},\ t_3^{(1)},\ l \ast \frac{1}{\theta_0} q^{(1)} + \frac{\alpha}{\theta_0} q^{(1)}\right),\\
\tau V &= \left(t_1^{(2)},\ t_2^{(2)},\ t_3^{(2)},\ l \ast \frac{1}{\theta_0} q^{(2)} + \frac{\alpha}{\theta_0} q^{(2)}\right).
\end{aligned}
$$

In the case of the homogeneous boundary conditions, from (4.7) we get

$$
(4.9)\qquad (AU \otimes V) = (U \otimes AV)
$$

and we are lead to the following functional:

$$
\begin{aligned}
\mathcal{M}(U) = (AU \otimes U) - 2(U \otimes F) &= \int_V g \ast (C_{ijkl} u_{i,j} \ast u_{k,l} + \\
&+ 2a_{ij} u_{i,j} \ast \theta)\mathrm{d}x + 2\int_V l \ast \alpha a_{ij} u_{i,j} \ast \theta \mathrm{d}x + \int_V [\rho u_i \ast u_i - \\
&- g \ast (l \ast k_{ij} \theta_{,j} + \alpha k_{ij} \theta_{,j}) \ast \theta_{,i}]\mathrm{d}x - \int_V g \ast d\theta \ast \theta \mathrm{d}x - \\
&- \int_V l \ast h\theta \ast \theta \mathrm{d}x - \int_V l \ast \alpha d\theta \ast \theta \mathrm{d}x - \int_V \alpha h\theta \ast \theta \mathrm{d}x - \\
&- 2\int_V [\rho f_i \ast u_i - (g \ast W + l \ast \alpha W) \ast \theta]\, \mathrm{d}x.
\end{aligned}
\tag{4.10}
$$

Theorem 4.2. *Let $D \subset D_A$ be the set of all vectors $U = (u_1, u_2, u_3, \theta)$ which satisfy the homogeneous boundary conditions* (3.6) (3.7). *Then*

$$
(4.11)\qquad \delta \mathcal{M}(U) = 0 \ \textit{on}\ D,
$$

if and only if $U \subset D$ satisfies the equation (4.4).

The condition (4.6) is a natural condition [3], [8].

Let us consider now the case of nonhomogeneous boundary conditions. We assume that $\Psi \subset D_A$ satisfies the boundary conditions

$$
(4.12)\qquad
\begin{aligned}
&\Psi_i = \tilde{u}_i \text{ on } \Sigma_1 \times [0, \infty), \quad \Theta = \tilde{\theta} \text{ on } \Sigma_3 \times [0, \infty),\\
&t_i(\psi) = \tilde{t}_i \text{ on } \Sigma_2 \times [0, \infty), \quad q(\psi) = \tilde{q} \text{ on } \Sigma_4 \times [0, \infty),
\end{aligned}
$$

where $\Psi = (\Psi_1, \Psi_2, \Psi_3, \Theta)$.

We introduce the vector $\mathcal{W}$ by relation $\mathcal{W} = U - \Psi$. Then $\mathcal{W}$ satisfies the equation

$$
(4.13)\qquad A\mathcal{W} = F - A\Psi
$$

and the homogeneous conditions (4.5), (4.6). In this case we are lead to the functional:

$$\Delta(\mathcal{W}) = \Delta(U - \Psi) = (A\mathcal{W}\otimes U) - 2(\mathcal{W}\otimes(F - A\Psi)) = (AU\otimes U) + \tag{4.14}$$
$$+(A\psi\otimes U) - (AU\otimes\psi) - 2(U\otimes F) + 2(\psi\otimes F) - (A\psi\otimes\psi).$$

Using (4.7), (4.10), (4.6) (4.5), (4.12) and neglecting inessential terms it follows that the functional (4.14) can be replaced by the the following:

$$\Gamma(U) = \int_V [g \ast (C_{ijkl}\, u_{k,l} \ast u_{i,j} + 2a_{ij}\, u_{i,j} \ast \theta) + l \ast 2\alpha a_{ij} u_{i,j} \ast \theta]\mathrm{d}x +$$
$$+ \int_V \rho u_i \ast u_i - g \ast (l \ast k_{ij}\, \theta_{,j} + \alpha k_{ij}\, \theta_{,j}) \ast \theta_{,i} -$$
$$- (g \ast d\theta + l \ast \alpha d\theta + l \ast h\theta + \alpha h\theta) \ast \theta - \int_{\Sigma_1} g \ast u_i \ast \tilde{t}_i \mathrm{d}\bar{x} - \tag{4.15}$$
$$- \int_{\Sigma_4} q \ast \frac{1}{\theta_0} (\alpha\tilde{q} + l \ast \tilde{q}) \ast \theta \mathrm{d}\Sigma - 2\int_V u_i \ast \rho f_i dx + 2\int_V (g \ast w + l \ast \alpha w) \ast \theta \mathrm{d}x.$$

Thus, taking into account that (4.6) is a natural condition, we have

Theorem 4.3. *Let $S \subset D_A$ be the set of all vectors U which satisfy the boundary conditions* (4.5). *Then,*

$$\delta\Gamma(U) = 0 \tag{4.16}$$

if, and only if $U \subset S$ is a solution of the mixed problem.

For $a_{ij} = -\beta_{ij}$, $a = \alpha = h = 0$, we find the reciprocity relation and the variational principle from [3].

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UNELE TEOREME ÎN O TEORIE GENERALIZATĂ A TERMOELASTICITĂŢII LINIARE

(Rezumat)

Teoria generalizată a termoelasticităţii liniare a fost elaborată de A. E. Green şi K. A. Lindsay, cu ajutorul inegalităţii producerii entropiei, propusă de A. E. Green şi N. Laws. În această lucrare se deduce o teoremă de reciprocitate, care duce la o teoremă variaţională de tip Gurtin, folosind metoda dată de D. Ieşan. Teorema de reciprocitate se deduce fără utilizarea transformatei Laplace iar principiul variaţional se stabileşte pentru problema cu valori iniţiale şi la limită, neomogene.

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STRUCTURI APROAPE PRODUS PE SPAŢII CU METRICI RELATIVISTE

DE

ELENA VAMANU

1. Introducere

A. Fie

$$R_{\alpha\beta} - \frac{1}{2} R g_{\alpha\beta} = \varkappa T_{\alpha\beta}, \quad (\alpha, \beta = 0, 1, 2, 3) \tag{1}$$

ecuaţiile gravitaţionale ale lui Einstein din teoria relativităţii, unde: $g_{\alpha\beta}$ este tensorul metric al unui spaţiu Riemann cu patru dimensiuni, V_4; $R_{\alpha\beta}$ — tensorul Ricci al acestui spaţiu, R — invariantul Ricci; $T_{\alpha\beta}$ — tensorul energie-impuls, iar $\varkappa$ o constantă universală.

Tensorul energie-impuls satisface condiţia de conservare

$$\nabla_\alpha T^\alpha_\beta = 0, \tag{2}$$

∇ fiind simbolul diferenţierii covariante în raport cu conexiunea spaţiului V_4 dată de simbolurile lui Christoffel.

Presupunînd tensorul $T_{\alpha\beta}$ cunoscut din consideraţii fizice, (1) reprezintă un sistem de ecuaţii cu derivate parţiale de ordin 2 în necunoscutele $g_{\alpha\beta}$. Sistemul (1) este în general greu de rezolvat şi se cunosc soluţii ale acestui sistem numai în cazuri speciale [1]...[3], [5], [6]. Odată $g_{\alpha\beta}$ determinate, este cunoscută metrica ds^2 a spaţiului V_4 şi deci toate proprietăţile geometrice ale lui V_4 care sînt şi proprietăţi fizice.

În această Notă se dă o soluţie a ecuaţiilor lui Einstein determinată de anumite condiţii impuse tensorului electromagnetic al lui Maxwell şi tensorului $T_{\alpha\beta}$ şi se studiază cîteva proprietăţi geometrice ale acestui model.

2. O soluţie a ecuaţiilor lui Einstein

B. Fie V_4 un spaţiu Riemann cu 4 dimensiuni a cărui metrică o presupunem statică şi cu simetrie sferică. În coordonate polare ea are forma

$$ds^2 = e^{2n} c^2 dt^2 - e^{2l} dr^2 - r^2(d\theta^2 + \sin^2\theta \, d\varphi^2), \tag{3}$$

unde n şi l sînt funcţii numai de r [3].

Din (3) rezultă că singurele componente nenule ale tensorului $g_{\alpha\beta}$ şi ale coeficienţilor de conexiune $\Gamma^\alpha_{\beta\gamma}$ daţi de simbolurile lui Christoffel sînt:

$$
(4)\quad \begin{cases} g_{00} = e^{2n}, \quad g_{11} = -e^{2l}, \quad g_{22} = -r^2, \quad g_{33} = -r^2 \sin^2\theta, \\ g^{00} = e^{-2n}, \quad g^{11} = -e^{-2l}, \quad g^{22} = -\dfrac{1}{r^2}, \quad g^{33} = -\dfrac{1}{r^2 \sin^2\theta}, \end{cases}
$$

$$
\begin{cases} \Gamma^0_{01} = n', \quad \Gamma^1_{00} = n' e^{2n-2l}, \quad \Gamma^1_{11} = l', \quad \Gamma^1_{22} = -re^{-2l}, \quad \Gamma^1_{33} = -re^{-2l}\sin^2\theta, \\ \Gamma^2_{12} = \Gamma^3_{13} = \dfrac{1}{r}, \quad \Gamma^2_{33} = -\sin\theta\cos\theta, \quad \Gamma^3_{23} = \operatorname{ctg}\theta, \end{cases}
$$

unde „ ' " însemană derivata în raport cu r.

Considerăm tensorul Ricci $R_{\alpha\beta}$,

$$
R_{\alpha\beta} = \Gamma^{\tau}_{\alpha\beta.\tau} - \Gamma^{\tau}_{\alpha\tau.\beta} + \Gamma^{\tau}_{\delta\tau}\Gamma^{\delta}_{\alpha\beta} - \Gamma^{\tau}_{\delta\beta}\Gamma^{\delta}_{\alpha\tau}.
$$

Singurele componente nenule ale tensorului $R_{\alpha\beta}$ sînt:

$$
(5)\quad \begin{cases} R_{00} = \left(n'' + n'^2 - l'n' + 2\dfrac{n'}{r}\right)e^{2n-2l}, \\ R_{11} = -n'' - n'^2 + l'n' + 2\dfrac{l'}{r}, \\ R_{22} = 1 + e^{-2l}(rl' - rn' - 1), \\ R_{33} = R_{22}\sin^2\theta. \end{cases}
$$

Invariantul Ricci R e dat de

$$
(6)\quad R = R_{\alpha\beta}g^{\alpha\beta} = 2e^{-2l}\left[n'' + n'^2 - n'l' + \frac{2(n'-l')}{r} + \frac{1}{r^2}\right] - \frac{2}{r^2}.
$$

În aceste condiţii ecuaţiile lui Einstein (1) devin:

$$
(7)\quad \begin{cases} e^{2n-2l}\left(\dfrac{2l'}{r} - \dfrac{1}{r^2}\right) + \dfrac{1}{r^2}e^{2n} = \varkappa T_{00}, \\ \dfrac{2n'}{r} + \dfrac{1}{r^2} - \dfrac{1}{r^2}e^{2l} = \varkappa T_{11}, \\ r^2 e^{-2l}\left(n'' + n'^2 - l'n' + \dfrac{n'-l'}{r}\right) = \varkappa T_{22}, \end{cases}
$$

cu necunoscutele $l = l(r)$ şi $n = n(r)$.

Pentru a determina l şi n din (7), trebuie să cunoaştem $T_{\alpha\beta}$. Se ştie că tensorul energie—impuls $T_{\alpha\beta}$ cuprinde un termen de origine electromagnetică reprezentat prin tensorul lui Maxwell $\tau_{\alpha\beta}$ şi un termen care dă distribuţia de masă reprezentat prin tensorul material $M_{\alpha\beta}$ [1], [3],

$$
(8)\quad T_{\alpha\beta} = \tau_{\alpha\beta} + M_{\alpha\beta}.
$$

În continuare, vom determina o soluţie a ecuaţiilor lui Einstein, pentru $T_{\alpha\beta}$ de forma (8), în următoarele condiţii:

I. Este cunoscut tensorul lui Maxwell $\tau_{\alpha\beta}$ şi în plus, acesta determină pe V_4 o structură aproape produs.

II. Dacă S_0, S_1, S_2, S_3 sînt valorile proprii ale tensorului $T_{\alpha\beta}$, acestea satisfac condiţia

$$S_0 = S_1 + S_2 + S_3.$$

Într-un sistem de coordonate polare, singurele componente nenule ale tensorului $\tau_{\alpha\beta}$ sînt [3]

$$(9) \qquad \begin{cases} \tau_{00} = \dfrac{e^2}{2r^4} e^{-2l}, & \tau_{11} = -\dfrac{e^2}{2r^4} e^{-2n}, \\ \tau_{22} = \dfrac{e^2}{2r^2} e^{-2n-2l}, & \tau_{33} = \tau_{22} \sin^2 \theta, \end{cases}$$

e fiind o constantă ce caracterizează particola ce crează cîmpul electromagnetic. Cum $\tau_{\alpha\beta}$ determină pe V_4 o structură aproape produs avem

$$(10) \qquad \tau^{\alpha}_{\beta} \tau^{\beta}_{\gamma} = \delta^{\alpha}_{\gamma}.$$

Condiţia (10) impusă tensorului $\tau_{\alpha\beta}$ conduce la relaţia

$$(11) \qquad e^{2n+2l} = \frac{e^2}{2r^4},$$

are, prin derivare, dă

$$(12) \qquad l' + n' = -\frac{2}{r}.$$

Valorile proprii ale tensorului $T_{\alpha\beta}$ sînt soluţiile ecuaţiei caracteristice

$$(13) \qquad \det (T_{\alpha\beta} - s g_{\alpha\beta}) = 0.$$

Condiţia II impusă valorilor proprii ale tensorului $T_{\alpha\beta}$ dă:

$$(14) \qquad T_{00} + T_{11} e^{2n-2l} \frac{2T_{22}}{r^2} e^{2n} = 0.$$

Din (7), (12), (14) şi condiţia de conservare (2) vom determina tensorii $g_{\alpha\beta}$ şi $T_{\alpha\beta}$.

Condiţia de conservare (2), scrisă pe larg, devine:

$$(15) \qquad (T^1_1)' + n' T^0_0 + \frac{2}{r}(T^1_1 - T^2_2) = 0.$$

Introducînd în (7) T_{00} dat de (14) obţinem:

$$(16)\quad \begin{cases} \dfrac{2l'}{r} - \dfrac{1}{r^2} + \dfrac{1}{r^2} e^{2l} = -\varkappa T_{11} - \dfrac{2\varkappa}{r^2} e^{2l} T_{22}, \\ \dfrac{2n'}{r} + \dfrac{1}{r^2} - \dfrac{1}{r^2} e^{2l} = \varkappa T_{11}, \\ r^2 e^{-2l}\left(n'' + n'^2 - l'n' + \dfrac{n' - l'}{r}\right) = \varkappa T_{22}. \end{cases}$$

Adunînd primele două ecuaţii din (16) şi ţinînd cont de (2), rezultă

$$(17)\qquad T_{22} = \frac{2}{\varkappa} e^{-2l}.$$

Condiţia de conservare (15) cu T_{00} dat de (14), T_{22} de (17) şi T_{11} de a doua ecuaţie din (16), conduce la

$$(18)\qquad n' = 0.$$

Cum $n' + l = -\dfrac{2}{r}$, rezultă

$$(19)\qquad l' = -\frac{2}{r}.$$

Din (18) şi (19) se determină n şi l şi avem

$$(20)\qquad g_{00} = e^{2n} = \beta^2, \qquad g_{11} = -e^{2l} = -\frac{e^2}{2\beta^2 r^4},$$

β fiind o constantă de integrare.

Deci, soluţia ecuaţiilor lui Einstein ne conduce la metrica relativistă

$$(21)\qquad ds^2 = \beta^2 (dx^0)^2 - \frac{e^2}{2\beta^2 r^4} dr^2 - r^2(d\theta^2 + \sin^2\theta d\varphi^2),$$

unde $x^0 = ct$, c fiind viteza luminii şi t timpul.

Din (14), (16) şi (17) se determină imediat şi componentele tensorului energie—impuls $T_{\alpha\beta}$ căruia îi corespunde soluţia (20):

$$(22)\quad \begin{cases} T_{00} = \dfrac{\beta^2(e^2 - 10\,\beta^2 r^4)}{e^2 r^2}, & T_{11} = \dfrac{2\beta^2 r^4 - e^2}{2\varkappa\,\beta^2 r^6}, \\ T_{22} = \dfrac{4\beta^2 r^4}{\varkappa e^2}, & T_{33} = T_{22}\sin^2\theta. \end{cases}$$

Valorile proprii ale tensorului $T_{\alpha\beta}$ sînt

$$(23)\qquad S_0 = \frac{e^2 - 10\,\beta^2 r^4}{\varkappa e^2 r^2}, \quad S_1 = \frac{e^2 - 2\beta^2 r^4}{\varkappa e^2 r^2}, \quad S_2 = S_3 = -\frac{4\beta^2 r^2}{\varkappa e^2}.$$

Dacă notăm

$$(24)\qquad S_0 = \rho, \quad S_1 = -p_1, \quad S_2 = -p_2, \quad S_3 = -p_3,$$

unde după cum se ştie, ρ se numeşte densitatea distribuţiei dată de $T_{\alpha\beta}$, iar p_1, p_2, p_3 se numesc presiuni parţiale, atunci condiţia II impusă tensorului $T_{\alpha\beta}$ revine la

$$\rho + p_1 + 2p_2 = 0. \tag{25}$$

Cunoscînd tensorul energie—impuls $T_{\alpha\beta}$ şi tensorul lui Maxwell $\tau_{\alpha\beta}$ din (8) putem determina şi distribuţia de masă $M_{\alpha\beta}$. Avem:

$$\left\{\begin{aligned} M_{00} &= \frac{\beta^2(e^2 - \varkappa e^2 r^2 - 10\beta^2 r^4)}{\varkappa e^2 r^2}, & M_{11} &= \frac{2\beta^2 r^4 + \varkappa e^2 r^2 - e^2}{2\varkappa\beta^2 r^6}, \\ M_{22} &= \frac{r^2(4\beta^2 r^2 - \varkappa e^2)}{\varkappa e^2}, & M_{33} &= M_{22}\sin^2\theta. \end{aligned}\right. \tag{26}$$

Se constată uşor că componentele tensorului $M_{\alpha\beta}$ satisfac relaţia

$$M_0^0 - M_1^1 - 2M_2^2 + 2 = 0. \tag{27}$$

3. Unele proprietăţi geometrice ale spaţiului V_4 cu metrica (21)

P_1. *În spaţiul V_4 există un cîmp de vectori v, determinat pînă la o translaţie în timp, a cărui derivată covariantă coincide cu tensorul lui Maxwell.*

În adevăr, în condiţiile de mai sus, cu τ_β^α dat, sistemul

$$\nabla_\beta v^\alpha = \tau_\beta^\alpha \tag{28}$$

este compatibil, admiţînd soluţia

$$v^0 = x^0 + \alpha, \qquad v^1 = -r, \qquad v^2 = v^3 = 0, \tag{29}$$

α fiind o constantă de integrare.

P_2. *Spaţiul Riemann V_4 admite o structură aproape produs de tipul celei studiate în* [4].

În adevăr, din faptul că τ_β^α determină pe V_4 o structură aproape produs şi din (28), rezultă:

$$\nabla_\alpha v^\tau . \nabla_\beta v^\rho g_{\tau\rho} = g_{\alpha\beta}, \tag{30}$$

$$\nabla_\alpha v_\beta - \nabla_\beta v_\alpha = 0, \tag{31}$$

condiţii care definesc structura aproape produs studiată în [4].

3.1. Consecinţe ale proprietăţilor P_1 şi P_2

1°. *Cîmpul de vectori v este un gradient.*

Această proprietate rezultă din (31).

2°. *Cîmpul de vectori v nu poate avea lungimea constantă.*

În adevăr, dacă notăm $l = g_{\alpha\beta}v^\alpha v^\beta$, avem

$$l = \beta^2(x^0 + \alpha)^2 - \frac{e^2}{2\beta^2 r^2}, \qquad \frac{\partial l}{\partial x^0} = 2\beta^2(x^0 + \alpha), \qquad \frac{\partial l}{\partial r} = \frac{e^2}{\beta^2 r^3} \tag{32}$$

şi măcar $\partial l/\partial r \neq 0$, deci lungimea lui v nu poate fi constantă.

3°. *Cîmpul de vectori v are divergenţa egală cu zero.*
În adevăr, avem

$$\operatorname{div} v = \nabla_\alpha v^\alpha = \tau^\alpha_\alpha.$$

Dar, o proprietate a tensorului lui Maxwell este tocmai $\tau^\alpha_\alpha = 0$. Deci $\operatorname{div} v = 0$.

3.2. Geodezicele spaţiului V_4

P_3. Facem următoarea schimbare de coordonate:

$$X^0 = \beta x^0, \quad R = \frac{e}{\sqrt{2}\,\beta}\frac{1}{r}, \quad \theta_1 = \frac{e}{\sqrt{2}\,\beta}\theta, \quad \varphi_1 = \frac{e}{\sqrt{2}\,\beta}\varphi. \tag{33}$$

În urma acestei schimbări, metrica spaţiului V_4 capătă forma[1])

$$ds^2 = (dx^0)^2 - dr^2 - \frac{1}{r^2}\left(d\theta^2 + \sin^2\frac{\sqrt{2}\,\beta}{e}\theta\, d\varphi^2\right). \tag{34}$$

Să determinăm geodezicele spaţiului V_4. Ecuaţiile lor sînt:

$$\frac{d^2x}{d\sigma^2} + \Gamma^\alpha_{\beta\gamma}\frac{dx^\beta}{d\sigma}\frac{dx^\gamma}{d\sigma} = 0, \quad (\alpha, \beta, \gamma = 0, 1, 2, 3).$$

Scrise pe larg ele devin:

$$\left\{\begin{aligned}
&\frac{d^2x^0}{d\sigma^2} = 0,\\
&\frac{d^2r}{d\sigma^2} + \frac{1}{r^3}\left(\frac{d\theta}{d\sigma}\right)^2 + \frac{1}{r^3}\sin^2\frac{\sqrt{2}\,\beta}{e}\theta\left(\frac{d\varphi}{d\sigma}\right)^2 = 0,\\
&\frac{d^2\theta}{d\sigma^2} - \frac{2}{r}\frac{dr}{d\sigma}\frac{d\theta}{d\sigma} - \sin\frac{\sqrt{2}\,\beta}{e}\theta\cos\frac{\sqrt{2}\,\beta}{e}\theta\left(\frac{d\varphi}{d\sigma}\right)^2 = 0,\\
&\frac{d^2\varphi}{d\sigma^2} - \frac{2}{r}\frac{dr}{d\sigma}\frac{d\varphi}{d\sigma} + 2\operatorname{ctg}\frac{\sqrt{2}\,\beta}{e}\theta\frac{d\theta}{d\sigma}\frac{d\varphi}{d\sigma} = 0.
\end{aligned}\right. \tag{35}$$

Alegînd sistemul de coordonate astfel ca condiţiile iniţiale să fie

$$\theta_0 = \frac{e}{2\sqrt{2}\,\beta}r, \quad \frac{d\theta_0}{d\sigma} = 0,$$

obţinem ecuaţiile geodezicelor în planul $\theta = \frac{e}{\sqrt{2}\,\beta}\frac{\pi}{2}$:

---

[1]) Pentru noile coordonate am folosit aceleaşi notaţii ca şi pentru cele vechi.

$$(36)\quad \begin{cases} x^0 = a\sigma + b, \\ r = a_1 \cos k\sigma + b_1 \sin k\sigma, \\ \varphi = \dfrac{k}{2}\left(a_1^2 + b_1^2\right)\sigma + \dfrac{1}{4}\left(a_1^2 - b_1^2\right)\sin 2k\sigma + a_1 b_1 \sin^2 k\sigma + a_2, \end{cases}$$

a, b, a_1, b_1, a_2, k fiind constante de integrare.

Geodezicele de lungime nulă, adică traiectoriile razelor luminoase, se obţin pentru următoarea relaţie între constantele de mai sus:

$$(37)\qquad a^2 - k^2(a_1^2 + b_1^2) = 0.$$

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STRUCTURES PRESQUE-PRODUIT SUR DES ESPACES À METRIQUES RELATIVISTES

(Résumé)

On détermine une solution des équations d'Einstein dans l'hypothèse que le tenseur électromagnétique de Maxwell détermine une structure presque-produit. On étudie ensuite quelques propriétés géométriques de ce modèle.

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UNELE OBSERVAŢII ASUPRA LEGILOR DE TRANSFORMARE DIN TERMODINAMICA RELATIVISTĂ

DE

CORNELIU D. CIUBOTARIU

1. Pe baza ipotezei că prima şi a doua lege a termodinamicii au aceeaşi formă în toate sistemele inerţiale, în anul 1908 Planck [1] a dedus următoarele formule de transformare pentru temperatură şi cantitatea de căldură :

$$T = T^0(1-\beta^2)^{\frac{1}{2}}, \quad \Delta Q = \Delta Q^0(1-\beta^2)^{\frac{1}{2}}, \quad \beta = \frac{v}{c}, \tag{1}$$

unde T^0 şi ΔQ^0 sînt temperatura şi, respectiv, cantitatea de căldură măsurate în referenţialul propriu. Aceste relaţii au fost unanim recunoscute valabile pînă în anul 1963 cînd Ott [2] a arătat că (1) poate conduce la unele rezultate eronate în calculul randamentului unei maşini termice relativiste, şi a propus legile de transformare :

$$T = T^0(1-\beta^2)^{-\frac{1}{2}}, \quad \Delta Q = \Delta Q^0(1-\beta^2)^{-\frac{1}{2}}. \tag{2}$$

Lucrarea lui Ott a fost urmată de un mare număr de articole contradictorii (vezi [3] şi [4] care cuprind o bibliografie completă), creîndu-se la un moment dat impresia că problema termodinamicii relativiste este încă destul de confuză.

În prezenta Notă, folosind modelul fluidului incoerent şi analizînd situaţiile la limită vom arăta că nici unul din tipurile de formule (1), (2), nu satisfac principiile relativităţii.

2. Referindu-ne numai la proprietăţile termodinamice macroscopice ale unui sistem, să presupunem că ecuaţia de stare în referenţialul propriu este $pV^0 = RT^0$. Deoarece p este un invariant (după cum rezultă din proprietăţile de transformare ale tensorului energie—impuls), un observator în mişcare va măsura $V = V^0(1-\beta^2)^{1/2}$ şi $T = T^0(1-\beta^2)^{1/2}$, în concordanţă cu relaţiile (1) ale lui Planck. Însă observatorul poate folosi ecuaţia echivalentă $E^0 = \frac{3}{2}RT^0$, şi cum $E = E^0(1-\beta^2)^{-1/2}$, rezultă pentru

T formula lui Ott. Prin urmare, ecuaţiile de stare din referenţialul propriu nu pot fi extinse în mod direct la referenţialul în mişcare. Încă un argument pentru această afirmaţie este existenţa relaţiei clasice $pV^0 = \frac{2}{3} E^0$ care nu poate fi folosită de un observator în mişcare deoarece V^0 şi E^0 posedă formule diferite de transformare.

Formularea legilor termodinamicii relativiste întîmpină astfel de dificultăţi deoarece însăşi construirea mecanicii sistemelor de particule în teoria relativităţii este o problemă complexă, energia şi impulsul avînd proprietăţi aditive doar în cazul sistemelor de particule care nu interacţionează.

3. Să considerăm un fluid incoerent şi să asociem noţiunea de temperatură stării de ordine a particulelor fluidului cu impuls şi energie determinate. Numim sistem propriu S^0, sistemul de referinţă legat de centrul de masă al fluidului. Impulsul şi energia totală ale fluidului faţă de acest sistem sînt:

$$P_i^0 = \Sigma m_{0\alpha}\, \gamma_\alpha^0\, v_{\alpha i}^0 = 0, \quad (i = x, y, z), \tag{3}$$

$$E^0 = \Sigma m_{0\alpha}\, \gamma_\alpha^0 c^2, \quad \gamma_\alpha^0 = \left(1 - \frac{v_\alpha^{02}}{c^2}\right)^{-\frac{1}{2}}, \tag{4}$$

unde $m_{0\alpha}$ este masa proprie a particulei α şi v_α^0 este viteza particulei α faţă de S^0. În raport cu un referenţial S care se deplasează cu viteza v în direcţia axei $-x$ a sistemului S^0, impulsul şi energia fluidului devin:

$$P_x = \frac{v}{c^2} E^0 \left(1 - \frac{v^2}{c^2}\right)^{-\frac{1}{2}}, \quad P_y = P_z = 0, \quad E = E^0 \left(1 - \frac{v^2}{c^2}\right)^{-\frac{1}{2}}. \tag{5}$$

Să admitem acum că am găsit un referenţial S' care se deplasează cu viteza v' în direcţia axei x a lui S, şi în raport cu care fluidul incoerent devine un curent incoerent de viteză u paralelă cu x. În S' avem

$$P'_x = M_0 \gamma u, \quad P'_y = P'_z = 0, \quad E' = M_0 \gamma c^2, \quad M_0 = \Sigma m_{0\alpha}. \tag{6}$$

În raport cu S relaţiile (6) devin:

$$\begin{aligned} P_x &= M_0 \gamma (u + v') \left(1 - \frac{v'^2}{c^2}\right)^{-\frac{1}{2}}, \quad P_y = P_z = 0, \\ E &= M_0 \gamma (c + v'u) \left(1 - \frac{v'^2}{c^2}\right)^{-\frac{1}{2}}, \quad \gamma = \left(1 - \frac{u^2}{c^2}\right)^{-\frac{1}{2}}. \end{aligned} \tag{7}$$

Din (5) şi (7) obţinem

$$v = (u + v') \left(1 + \frac{uv'}{c^2}\right)^{-1}, \tag{8}$$

adică legea relativistă de compunere a vitezelor. Prin urmare, există un referenţial inerţial în care se poate defini o distribuţie ordonată echivalentă

(aceiaşi E şi P) cu fluidul incoerent iniţial. Dacă temperatura este o mărime asociată stării de ordine a fluidului, atunci ea este o mărime care depinde de viteza observatorului.

4. Mariot [5], folosind unicitatea ecuaţiilor lui Maxwell pentru cîmpul electromagnetic singular, a arătat existenţa unui fluid de fotoni ale cărui traiectorii sînt geodezicele izotrope ale spaţiului-timp. În adevăr, tensorul energie-impuls al cîmpului electromagnetic singular poate fi pus sub forma

$$\tau_{\alpha\beta} = w l_\alpha l_\beta , \tag{9}$$

unde

$$l^\beta \nabla_\beta l^\alpha = a l^\alpha , \quad l^\alpha l_\alpha = 0, \tag{10}$$

iar w este densitatea de energie electromagnetică. Să definim pentru fluidul de fotoni cuadrivectorul temperatură $T_\alpha = T^0 l_\alpha$. Dacă impunem condiţia ca norma acestui vector să fie egală cu temperatura proprie invariantă T^0, în conformitate cu definiţia temperaturii relativiste a lui Arzelès [6], obţinem $T^0 = 0$, adică fluidul incoerent de fotoni are o distribuţie ordonată independentă de viteza observatorului. Acest rezultat este în concordanţă cu relaţia (8) conform căreia un fluid devine perfect ordonat pentru oricare din limitele $v \to c$, $u \to c$, $v' \to c$.

În concluzie, deşi formulele de transformare (2) ale lui Ott s-au dovedit utile într-o serie de consideraţii ale termodinamicii relativiste, ele nu îndeplinesc condiţia la limită ca un sistem cu viteza $v \to c$ să apară într-o stare ordonată pentru observatorul în repaus.

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SOME REMARKS CONCERNING THE TRANSFORMATION LAWS IN RELATIVISTIC THERMODYNAMICS

(Summary)

The generalization of thermodynamics to special relativity is discussed in the case of an incoherent fluid with the center-of-mass momentum as a thermodynamic variable.

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EXTENSION FORMULAS FOR SL(4, C): THE PRINCIPAL NONDEGENERATE SERIES

BY

GH. ZET

1. Introduction

In the last years, many applications of the unitary infinite-dimensional representations of noncompact Lie groups, have appeared. For applications in elementary particle physics, it is necessary to have the corresponding representations in a discrete basis (whose elements can be associated with particle states) and their reduction to the maximal compact subgroup [1]. These problems are completely solved for the groups semidirect product [2], [3]. In this paper we describe a method for solving these problems for the principal nondegenerate series of unitary irreducible representations of SL(4, C). The prinicpal degenerate series have been treated in previous works [4], [5].

The generators of SL(4, C) are written as polynomial functions (the extension formulas) of the generators of suitably chosen unitary representations of $SU(4)\times)T_{15}$ (semidirect product [4]). Since the representations of $SU(4)\times)\,T_{15}$ in a SU(4) basis are known [2], this provides a solution for principal nondegenerate series of SL(4, C).

The generators of SL(4, C) are made up from the generators M_α^β ($\alpha, \beta = 1, 2, 3, 4$ and $M_\alpha^\alpha = 0$) of its maximal compact subgroup SU(4) and from the „noncompact" generators $N_\alpha^\beta(\alpha, \beta = 1, 2, 3, 4$ and $N_\alpha^\alpha = 0)$. (The sum convention over repeated indices is made). The commutation relations in a unitary representation of SL(4, C) are:

$$
(1.1)\qquad \begin{cases} [M_\alpha^\beta, M_\mu^\lambda] = \delta_\mu^\beta M_\alpha^\lambda - \delta_\alpha^\lambda M_\mu^\beta, \\ [M_\alpha^\beta, N_\mu^\lambda] = \delta_\mu^\beta N_\alpha^\lambda - \delta_\alpha^\lambda N_\mu^\beta, \\ [N_\alpha^\beta, N_\mu^\lambda] = \delta_\alpha^\lambda M_\mu^\beta - \delta_\mu^\beta M_\alpha^\lambda, \end{cases}
$$

where δ_β^α is the Kronecker symbol. In addition, the followings Hermiticity properties are satisfied

$$(M^{\alpha}_{\beta})^{+} = M^{\beta}_{\alpha}, \quad (N^{\alpha}_{\beta})^{+} = N^{\beta}_{\alpha}. \tag{1.2}$$

We consider now the followings two subgroups of SL(4, C):

$$Z \ni z = \begin{pmatrix} 1 & 0 & 0 & 0 \\ \gamma & 1 & 0 & 0 \\ \beta & \alpha & 1 & 0 \\ \omega & \theta & \varphi & 1 \end{pmatrix}, \quad K \ni k = \begin{pmatrix} k_{11} & k_{12} & k_{13} & k_{14} \\ 0 & k_{22} & k_{23} & k_{24} \\ 0 & 0 & k_{33} & k_{34} \\ 0 & 0 & 0 & k_{44} \end{pmatrix}, \quad \det k = 1 \tag{1.3}$$

where α, β, γ, ω, θ, φ are arbitrary complex numbers. Then, almost every element $g \in SL(4, C)$ may be written in a unique way as the product of one element in Z and one element in K [6]:

$$g = kz, \quad k \in K, \quad z \in Z. \tag{1.4}$$

In particular, the element $zg \in SL(4, C)$ may be written as:

$$zg = kz', \quad z' \in Z. \tag{1.5}$$

The representations $g \to R(g)$ of the principal nondegenerate series are characterized by six parameters $(m_2, m_3, m_4, \rho_2, \rho_3, \rho_4)$ where m_2, m_3, m_4 are arbitrary integer numbers and ρ_2, ρ_3, ρ_4 arbitrary real numbers. The representations of this series are constructed in the Hilbert space $L^2(Z)$ of all functions $f(z)$ which are square-integrable. The action of the unitary operator $R(g)$ on a such function $f(z)$ is given by formula [6]:

$$(R_g f)(z) = \alpha(k) f(z'), \qquad f(z), f(z') \in L^2(Z), \tag{1.6}$$

where

$$\alpha(k) = |k_{22}|^{m_2 + i\rho_2 - 2} k_{22}^{-m_2} |k_{33}|^{m_3 + i\rho_3 - 4} k_{33}^{-m_3} |k_{44}|^{m_4 + i\rho_4 - 6} k_{44}^{-m_4} \tag{1.7}$$

and the variables α, β, γ, ω, θ, φ are determinated by (1.5).

Now, the formula (1.6) has the disadvantage that its right-hand side has a dependence on the continuous parameters ρ_2, ρ_3, ρ_4, when $g \in SU(4)$. Or, it is known that unitary irreducible representations of SU(4) are characterized by discrete indices only. The situation changes if we consider an equivalent representation $R'_g = SR_gS^{-1}$, where S is a transformation in $L^2(Z)$, defined as:

$$\begin{cases} (Sf)(z) = \exp[i\Phi(z)] f(z), \\ \Phi(z) = -\dfrac{\rho_2}{2} \ln A + \dfrac{1}{2}(\rho_2 - \rho_3) \ln B + \dfrac{1}{2}(\rho_3 - \rho_4) \ln C, \\ A = 1 + |\gamma|^2 + |\beta - \alpha\gamma|^2 + |\gamma\theta - \omega + \varphi(\beta - \alpha\gamma)|^2, \\ B = 1 + |\alpha|^2 + |\beta|^2 + |\alpha\varphi - \theta|^2 + |\beta\varphi - \omega|^2 + \\ \quad + |\beta\theta - \alpha\omega|^2, \\ C = 1 + |\omega|^2 + |\theta|^2 + |\varphi|^2. \end{cases} \tag{1.8}$$

Then, (1.6) can be rewritten as

$$(R'_g f)(z) = \exp \mathrm{i}[\Phi(z) - \Phi(z')]\,\alpha(k) f(z'). \tag{1.9}$$

In this representation, the generators M^β_α and N^β_α are given by formulas :

$$\begin{aligned} M^\beta_\alpha = \sum_{p=2}^{4} \Bigg[& \frac{m_p}{4}\delta^\beta_\alpha - \frac{m_p + 2(p-1)}{2}(3\delta^\beta_\alpha z^p_p - 3z^\beta_p z^p_\alpha + \\ & + z^\beta_\lambda z^\lambda_p z^p_\alpha) - \frac{m_p - 2(p-1)}{2}(3\delta^\alpha_p z^{p*}_\beta - 3z^{\alpha*}_p z^{p*}_\beta + z^{\alpha*}_\lambda z^{\lambda*}_p z^{p*}_\beta)\Bigg] + \\ & + (4\delta^\beta_\lambda z^\mu_\alpha - 6z^\beta_\lambda z^\mu_\alpha + 4z^\beta_\nu z^\nu_\lambda z^\mu_\alpha - z^\beta_\nu z^\nu_\sigma z^\sigma_\lambda z^\mu_\alpha)\frac{\partial}{\partial z^\mu_\lambda} - \\ & - (4\delta^\alpha_\lambda z^{\mu*}_\beta - 6z^{\alpha*}_\lambda z^{\mu*}_\beta + 4z^{\alpha*}_\nu z^{\nu*}_\lambda z^{\mu*}_\beta - z^{\alpha*}_\nu z^{\nu*}_\sigma z^{\sigma*}_\lambda z^{\mu*}_\beta)\frac{\partial}{\partial z^{\mu*}_\lambda}, \end{aligned} \tag{1.10}$$

$$N^\beta_\alpha = \rho_2 P^\beta_\alpha + \overline{N}^\beta_\alpha, \tag{1.11}$$

where

$$\begin{aligned} \overline{N}^\beta_\alpha = -\mathrm{i}\sum_{p=2}^{4} \Bigg[& \frac{p-1}{2}\delta^\beta_\alpha - \frac{m_p + 2(p-1)}{2}(3\delta^\beta_p z^p_\alpha - 3z^\beta_p z^p_\alpha + z^\beta_\lambda z^\lambda_p z^p_\alpha) + \\ & + \frac{m_p - 2(p-1)}{2}(3\delta^\alpha_p z^{p*}_\beta - 3z^{\alpha*}_p z^{p*}_\beta + z^{\alpha*}_\lambda z^{\lambda*}_p z^{p*}_\beta)\Bigg] - \\ & - \mathrm{i}(4\delta^\beta_\lambda z^\mu_\alpha - 6z^\beta_\lambda z^\mu_\alpha + 4z^\beta_\nu z^\nu_\lambda z^\mu_\alpha - z^\beta_\nu z^\nu_\sigma z^\sigma_\lambda z^\mu_\alpha)\frac{\partial}{\partial z^\mu_\lambda} - \\ & - \mathrm{i}(4\delta^\alpha_\lambda z^{\mu*}_\beta - 6z^{\alpha*}_\lambda z^{\mu*}_\beta + 4z^{\alpha*}_\nu z^{\nu*}_\lambda z^{\mu*}_\beta - z^{\alpha*}_\nu z^{\nu*}_\sigma z^{\sigma*}_\lambda z^{\mu*}_\beta)\frac{\partial}{\partial z^{\mu*}_\lambda} \end{aligned} \tag{1.12}$$

and

$$\begin{cases} P^\beta_\alpha = u_1 A^\beta_\alpha + s_1 B^\beta_\alpha + t_1 C^\beta_\alpha, \\ u_1 = 1, \quad s_1 = s = \dfrac{\rho_3}{\rho_2} - 1, \quad t_1 = t = \dfrac{\rho_4}{\rho_2} - 1; \end{cases} \tag{1.13}$$

$$\begin{cases} A^1_1 = \dfrac{1}{4} - \dfrac{1}{A}, \quad A^2_1 = \dfrac{\gamma}{A}, \quad A^3_1 = \dfrac{\beta - \alpha\gamma}{A}, \quad A^4_1 = -\dfrac{\gamma\theta - \omega + \varphi(\beta - \alpha\gamma)}{A}, \\ A^2_2 = \dfrac{1}{4} - \dfrac{|\gamma|^2}{A}, \quad A^3_2 = -\dfrac{\gamma^*(\beta - \alpha\gamma)}{A}, \quad A^4_2 = \dfrac{\gamma^*[\gamma\theta - \omega + \varphi(\beta - \alpha\gamma)]}{A}, \\ A^3_3 = \dfrac{1}{4} - \dfrac{|\beta - \alpha\gamma|^2}{A}, \quad A^4_3 = \dfrac{(\beta - \alpha\gamma)^*[\gamma\theta - \omega + \varphi(\beta - \alpha\gamma)]}{A}, \\ A^4_4 = \dfrac{1}{4} - \dfrac{|\gamma\theta - \omega + \varphi(\beta - \alpha\gamma)|^2}{A}, \quad A^\alpha_\alpha = 0; \end{cases} \tag{1.13'}$$

$$
(1.13'')\quad \left\{
\begin{aligned}
B_1^1 &= \frac{|\beta|^2 + |\beta\varphi - \omega|^2 + |\beta\theta - \alpha\omega|^2}{B} - \frac{1}{4} - \frac{|\omega|^2}{C}, \\
B_1^2 &= \frac{\alpha^*\beta + (\alpha\varphi - \theta)^*(\beta\varphi - \omega)}{B} - \frac{\omega\theta^*}{C}, \\
B_1^3 &= \frac{\beta - (\alpha\varphi - \theta)^*(\beta\theta - \alpha\omega)}{B} - \frac{\omega\varphi^*}{C}, \\
B_1^4 &= -\frac{(\beta\varphi - \omega) + \alpha^*(\beta\theta - \alpha\omega)}{B} - \frac{\omega}{C}, \\
B_2^2 &= \frac{|\alpha|^2 + |\alpha\varphi - \theta|^2 + |\beta\theta - \alpha\omega|^2}{B} - \frac{1}{4} - \frac{|\theta|^2}{C}, \\
B_2^3 &= \frac{\alpha + (\beta\varphi - \omega)^*(\beta\theta - \alpha\omega)^2}{B} - \frac{\theta\varphi^*}{C}, \\
B_2^4 &= \frac{-(\alpha\varphi - \theta) + \beta^*(\beta\theta - \alpha\omega)}{B} - \frac{\theta}{C}, \\
B_3^3 &= \frac{1 + |\alpha\varphi - \theta|^2 + |\beta\varphi - \omega|^2}{B} - \frac{1}{4} - \frac{|\varphi|^2}{C}, \\
B_3^4 &= \frac{\alpha^*(\alpha\varphi - \theta) + \beta^*(\beta\varphi - \omega)}{B} - \frac{\varphi}{C}, \\
B_4^4 &= \frac{1 + |\alpha|^2 + |\beta|^2}{B} - \frac{1}{4} - \frac{1}{C}, \quad B_\alpha^\alpha = 0;
\end{aligned}
\right.
$$

$$
(1.13''')\quad \left\{
\begin{aligned}
&C_1^1 = \frac{|\omega|^2}{C} - \frac{1}{4}, \quad C_1^2 = \frac{\omega\theta^*}{C}, \quad C_1^3 = \frac{\omega\varphi^*}{C}, \\
&C_1^4 = \frac{\omega}{C}, \quad C_2^2 = \frac{|\theta|^2}{C} - \frac{1}{4}, \\
&C_2^3 = \frac{\theta\varphi^*}{C}, \quad C_2^4 = \frac{\theta}{C}, \quad C_3^3 = \frac{|\varphi|^2}{C} - \frac{1}{4}, \\
&C_3^4 = \frac{\varphi}{C}, \quad C_4^4 = \frac{1}{C} - \frac{1}{4}, \quad C_\alpha^\alpha = 0.
\end{aligned}
\right.
$$

From the component of P_α^β that we have listed, the remaining ones can be obtained by using the relations (1.2). (The matrix elements of $z \in Z$ have been denoted by z_μ^λ).

Obviously, by contraction

$$\lim_{\rho_2 \to \infty} \frac{1}{\rho_2} N_\alpha^\beta = P_\alpha^\beta , \tag{1.14}$$

we obtain a class of unitary representations of $SU(4)\times)T_{15}$.

2. The Extension Formulas

The generators N_α^β can be written as polynomial functions (the extension formulas) of the generators M_α^β and P_α^β of $SU(4)\times)T_{15}$. To obtain the extension formulas, first we consider the expressions

$$Q_\alpha^\beta = P_\gamma^\beta P_\alpha^\gamma - \frac{1}{4} \delta_\alpha^\beta P_\mu^\lambda P_\lambda^\mu , \tag{2.1}$$

$$R_\alpha^\beta = P_\gamma^\beta P_\sigma^\gamma P_\alpha^\sigma - \frac{1}{4} \delta_\alpha^\beta P_\mu^\lambda P_\nu^\mu P_\lambda^\nu . \tag{2.2}$$

Now, from (1.13) it follows that

$$\begin{cases} Q_\alpha^\beta = u_2 A_\alpha^\beta + s_2 B_\alpha^\beta + t_2 C_\alpha^\beta , \quad s_2 = \frac{1}{2}(s_1^2 - s_1 t_1 + s_1 u_1), \\ t_2 = \frac{1}{2}(t_1^2 - s_1 t_1 + t_1 u_1), \quad u_2 = -\frac{1}{2}(u_1^2 + s_1 u_1 + t_1 u_1) ; \end{cases} \tag{2.3}$$

$$\begin{cases} R_\alpha^\beta = u_3 A_\alpha^\beta + s_3 B_\alpha^\beta + t_3 C_\alpha^\beta , \\ s_3 = \frac{1}{4}(2 s_2 s_1 - s_2 t_1 + s_2 u_1 - t_2 s_1 + u_2 s_1), \\ t_3 = \frac{1}{4}(- s_2 t_1 - t_2 s_1 + 2 t_2 t_1 + t_2 u_1 + u_2 t_1), \\ u_3 = \frac{1}{4}(- s_2 u_1 - t_2 u_1 - u_2 s_1 - u_2 t_1 - 2 u_2 u_1). \end{cases} \tag{2.4}$$

Using these formulas, we try to writte the $\overline{N}_\alpha^\beta$ as a linear combination of M_α^β and P_α^β:

$$\begin{aligned} \overline{N}_\alpha^\beta = {} & \mathrm{i}a(PM - MP)_\alpha^\beta + \mathrm{i}b(QM - MQ)_\alpha^\beta + \mathrm{i}c(RM - MR)_\alpha^\beta + \\ & + \mathrm{i}d(QMP - PMQ)_\alpha^\beta + \mathrm{i}e(RMP - PMR)_\alpha^\beta + \mathrm{i}f(RMQ - QMR)_\alpha^\beta . \end{aligned} \tag{2.5}$$

Comparision of (2.5) and (1.12) gives the following sistem of linear equations for coefficients a, b, c, d, e and f:

$$
(2.6)\quad \begin{cases}
u_1a + u_2b + u_3c = \dfrac{1}{2}\,, \\[2ex]
s_1a + s_2b + s_3c = \dfrac{1}{2}\,, \quad t_1a + t_2b + t_3c = 1, \\[2ex]
(s_2u_1 - s_1u_2)d + (s_3u_1 - s_1u_3)e + (s_3u_2 - s_2u_3)f = 1, \\
(s_2t_1 - s_1t_2)d + (s_3t_1 - s_1t_3)e + (s_3t_2 - s_2t_3)f = -1, \\
(t_2u_1 - u_2t_1)d + (t_3u_1 - u_3t_1)e + (u_2t_3 - u_3t_2)f = 1.
\end{cases}
$$

The solution of this sistem is

$$
(2.7)\quad \begin{cases}
a = \dfrac{2q_3 - p_3 - r_3}{2(q_1r_2 - q_2r_1)}\,, \quad b = \dfrac{-2q_2 + p_2 + r_2}{2(q_1r_2 - q_2r_1)}\,, \quad c = \dfrac{2q_1 - p_1 - r_1}{2(q_1r_2 - q_2r_1)}\,, \\[2ex]
d = \dfrac{(p_2r_3 - p_3r_2) + (q_2r_3 - q_3r_2) + (q_2p_3 - p_2q_3)}{q_1(p_2r_3 - p_3r_2) - q_2(p_1r_3 - p_3r_1) + q_3(p_1r_2 - p_2r_1)}\,, \\[2ex]
e = \dfrac{-(p_1r_3 - p_3r_1) - (q_1r_3 - q_3r_1) - (q_1p_3 - q_3p_1)}{q_1(p_2r_3 - p_3r_2) - q_2(p_1r_3 - p_3r_1) + q_3(p_1r_2 - p_2r_1)}\,, \\[2ex]
f = \dfrac{(p_1r_2 - p_2r_1) + (q_1r_2 - q_2r_1) + (q_1p_2 - q_2p_1)}{q_1(p_2r_3 - p_3r_2) - q_2(p_1r_3 - p_3r_1) + q_3(p_1r_2 - p_2r_1)}\,,
\end{cases}
$$

where

$$
(2.8)\quad \begin{cases}
p_1 = (s_2t_1 - s_1t_2), \quad p_2 = (s_3t_1 - s_1t_3), \quad p_3 = (s_3t_2 - s_2t_3), \\
q_1 = (s_2u_1 - s_1u_2), \quad q_2 = (s_3u_1 - s_1u_3), \quad q_3 = (s_3u_2 - s_2u_3), \\
r_1 = (t_2u_1 - u_2t_1), \quad r_2 = (t_3u_1 - u_3t_1), \quad r_3 = (u_2t_3 - u_3t_2).
\end{cases}
$$

Therefore, the extension formulas for the principal nondegenerate series, of SL(4, C) are

$$
(2.9)\quad \begin{aligned}
N^{\beta}_{\alpha} = \rho_2 P^{\beta}_{\alpha} + \mathrm{i}[&a(PM - MP)^{\beta}_{\alpha} + b(QM - MQ) + c(RM - MR)^{\beta}_{\alpha} + \\
&+ d(QMP - PMQ)^{\beta}_{\alpha} + e(RMP - PMR)^{\beta}_{\alpha} + f(RMQ - QMR)^{\beta}_{\alpha}],
\end{aligned}
$$

where the coefficients $a...f$ are given by (2.7). Because the representations of SU(4)$\times$)T_{15} can be constructed in a SU(4) basis, we can calculate, via expansion formulas, the matrix elements of M^{β}_{α} and N^{β}_{α} in a SU(4) basis.

3. Generalization to SL(n,C) Case

We consider the combinations

$$
(3.1)\quad (Q_i)^{\beta}_{\alpha} = P^{\beta}_{\gamma_1} P^{\gamma_1}_{\gamma_2} \dots P^{\gamma_i}_{\alpha} - \frac{1}{n}\,\delta^{\beta}_{\alpha} P^{\lambda}_{\mu_1} P^{\mu_1}_{\nu_1} \dots P^{\mu_i}_{\lambda}\,, \quad i = -1, 0, 1, 2, \dots, n-2,
$$

where by definition $(Q_{-1})^{\beta}_{\alpha} = \delta^{\beta}_{\alpha}$ and $(Q_0)^{\beta}_{\alpha} = P^{\beta}_{\alpha}$. Then, the extension formulas for the principal nondegenerate series of SL(n, C) have the form:

$$N^{\beta}_{\alpha} = \rho_2 P^{\beta}_{\alpha} + i \sum_{j>k} a_{jk} (Q_j M Q_k - Q_k M Q_j)^{\beta}_{\alpha}, \tag{3.2}$$

$$j, k = -1, 0, 1, 2, \ldots, (n-2)$$

The $\frac{n(n-1)}{2}$ coefficients a_{jk} are determined by identification of (3.2) with the expression of generators N^{β}_{α} in this representation; this determination will be taken up elsewhere.

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FORMULE DE EXTENSIUNE PENTRU SL(4, C): SERIILE PRINCIPALE NEDEGENERATE

(Rezumat)

Se dă o metodă de reprezentare într-o bază discretă şi de reducere la subgrupul compact maxim a seriilor principale nedegenerate ale reprezentărilor ireductibile unitare ale grupului SL(4, C), grup care intervine în teoria particulelor elementare.

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SUR LA CONDUCTION ÉLECTRIQUE DES FERRITES QUI CONTIENNENT DU CUIVRE

PAR

N. REZLESCU, DORIN CONDURACHE et EMIL LUCA

Pour les ferrites qui contiennent du cuivre nous avons constaté un mécanisme de conduction électrique différent par rapport aux majorité des ferrites spinelles, qui manifestent une conduction de type n, réalisé par échange électronique $Fe^{2+} \rightleftarrows Fe^{3+}$ [1].

Dans le présent ouvrage nous donnons quelques résultats expérimentaux en ce qui concerne le mécanisme de conduction électrique des ferrites avec du cuivre et l'anomalie dans le point de Curie de l'effet Seebeck pour ces ferrites.

Les mesures ont été faites sur les ferrites de composition stoechio métrique, préparés par la technologie connue des oxydes. Les échantillons ont été frittés à la température de 1 200°C durant 4 heures, pui refroidis rapidement dans l'air. Les analyses faites après le frittage sur les échantillons ont indiqué l'existence d'une seule phase, spinelle.

Le type de mécanisme de conduction électrique a été déterminé par le signe de pouvoir thermo-électrique (effet Seebeck). Nous avons déterminé aussi la variation de l'effet Seebeck en fonction de température pour des ferrites qui contiennent du cuivre. Mais nous donnons seulement les résultats obtenus sur deux échantillons : $Cu_{0,6}Zn_{0,4}Fe_2O_4$ (fig. 1), avec le point de Curie $T_c = 136°C$ et $Cu_{0,3}Mn_{0,3}Zn_{0,4}Fe_2O_4$ (fig. 2), avec $T_c = 158°C$. Nous avons mesuré spécialement les échantillons qui contiennent aussi du zinc parce que ces échantillons ont les points de Curie situés aux températures acceptables pour les mesures de l'effet Seebeck autour du point de Curie.

Après la détermination de la courbe de variation du coefficient Seebeck en fonction de la température sur l'échantillon refroidi rapidement, nous avons traité cet échantillon dans une atmosphère d'oxygène à la température de 1 000°C durant 2 heures. Par ce traitement le point de Curie a été déplacè vers les températures plus élevées : à 231°C pour Cu-Zn ferrite et à 199°C pour Cu-Mn-Zn ferrite [2].

On observe sur les fig.1 et 2 que les courbes $\theta = f(1/T)$ présentent des points d'anomalie dans les domaines étroits de température dans lesquels sont situés aussi les points de Curie des échantillons respectifs ; en

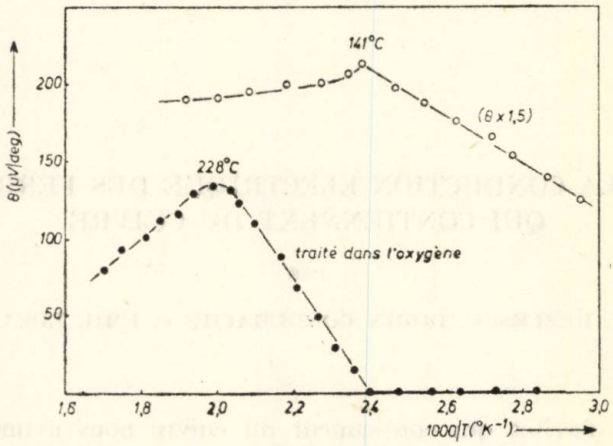

Fig. 1

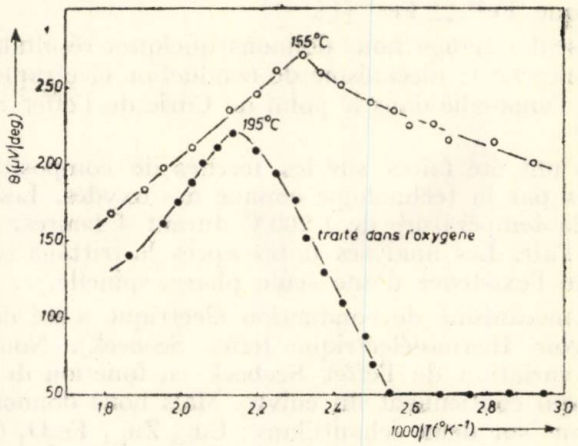

Fig. 2

général, jusqu'à la température de Curie θ croît et puis il diminue. Un changement du signe de la pente de la courbe $\theta = f(1/T)$ suppose une modification du mécanisme de conduction. Ce fait indique une forte influence de l'ordre ferrimagnétique sur le processus de conduction électrique dans le ferrite.

Mais le passage par le point de Curie ne modifie pas le signe de la conduction. Ainsi, nous avons constaté que les ferrites avec du cuivre manifestent une conduction de type *p*, par lacunes. L'origine de cette conduction électrique comporte des discussions.-

Avec les notations suivantes : n et p — concentrations des porteurs positifs et negatifs, μ_n et μ_p — leurs mobilités, 2Δ — zone interdite entre les niveaux inférieurs et supérieurs, f — position du niveau Fermi par rapport au milieu de la zone interdite et q_n et q_p — fractions d'énergie d'activation déterminées par les porteurs n et p, nous pouvons écrire le coefficient Peltier sous la forme [3]

$$\Pi = \theta T = \frac{p\mu_p(\Delta + f + q_p) - n\mu_n(\Delta - f + q_n)}{ne\mu_n + pe\mu_p}, \tag{1}$$

où e est la charge des porteurs n et p.

Evidément, le signe du coefficient Seebeck θ dépend du rapport des termes du numérateur de l'équation précédente. Si les porteurs positifs p prédominent, θ est positif et la conductibilité électrique est donnée par

$$\sigma = pe\mu_p . \tag{2}$$

On sait [4], [5] que dans les ferrites qui contiennent du cuivre il y a une évidente tendance de réduction d'ion Cu^{2+} après 900—950°C pendant le frittage. Les ions Cu^{1+} apparus dans ces ferrites modifient fortement leurs propriétés électriques et magnétiques [6], [7]. Il est raisonable de lier l'éxistence des porteurs p dans les ferrites qui contiennent du cuivre avec l'apparition des ions Cu^{1+} pendant le frittage [8]. En supposant qu'à la formation de chaque paire d'ions Cu^{1+} , un atom d'oxygène est éliminé, une quantité égale de ions métaliques doivent occuper les sites interstitiels, très nombreaux dans un réseau spinelle. Mais les ions Cu^{1+} sont des porteurs de type p ; ainsi seulement leur présence n'éxplique pas la conduction de type n des ferrites avec copper. Mais, en supposant que les ions métaliques interstitiels se comportent comme centres accepteurs, alors les porteurs p on peut les identifier comme lacunes sur les ions d'oxygène.

Dans nos ferrites il y a certainement aussi des porteurs n, les ions Fe^{2+} , étant donné surtout l'existence du zinc dans ces ferrites [1]. Parce que dans les ferrites qui contiennent du cuivre se manifeste seulement la conduction p, on peut supposer que cette conduction est plus importante que la conduction n pour ces ferrites.

On voit dans les fig. 1 et 2 qu'après le traitement dans l'oxygène le coefficient Seebeck diminue. Ce fait s'explique par la diminution du nombre des porteurs (a lieu le passage $Cu^{1+} \rightarrow Cu^{2+}$). Même si le passage $Fe^{2+} \rightarrow$ $\rightarrow Fe^{3+}$ a aussi lieu pendant ce traitement, la conduction p reste plus importante.

Aussi, on observe (fig. 1 et 2) qu'on peut mesurer le coefficient Seebeck seulement au delà d'une température assez élevée, lorsque la mobilité des porteurs est grande. Ce fait-ci est déterminé aussi par la diminution du nombre des porteurs de charge après le traitement dans l'oxygène.

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ASUPRA CONDUCŢIEI ELECTRICE A FERITELOR CARE CONŢIN CUPRU

(Rezumat)

Prin cercetări de forţă termoelectromotoare (efect Seebeck) se arată că în feritele care conţin cupru conducţia se face în principal prin purtători *p*, spre deosebire de majoritatea feritelor spinelice, care manifestă o conducţie *n*. Se explică apariţia purtătorilor *p* în feritele ce conţin cupru.

Efectul Seebeck are o anomalie la trecerea prin punctul Curie; în acest punct panta curbei $\theta = f(1/T)$ îşi schimbă semnul.

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